AN INTER-CONTINENTAL TIE DETERMINED FROM THE OBSERVATION OF AN ARTIFICIAL SATELLITE

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AN INTER-CONTINENTAL TIE DETERMINED FROM THE OBSERVATION OF AN ARTIFICIAL SATELLITE

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PREFACE

The application of artificial earth satellites in geodesy can be either dynamic or geometric.

In the dynamic approach, an evaluation of the perturbations in the orbital motion of a satellite can furnish data from which the coefficients of the spherical harmonic expansion of the potential function can be determined. Using this approach, valuable information on the gravitational field of the earth can be obtained.

In the geometric treatment, the artificial earth satellite is used as a reference point or a triangulation station in the sky. In this way the positions of observation stations on the surface of the earth can be determined. Some aspects of the geometric approach will be discussed in this paper.

The purpose of this thesis is to serve as a demonstration of the material studied during its preparation. I would like to emphasize that the reader should not construe the methods and procedures used herein as being of a standard type because of the following reasons:

1. The computed orbit from three observations at a given station at short intervals is assumed to be correct and accurate;

2. It is assumed that the observed coordinates of the satellite taken from Smithsonian Institution Astrophysical Observatory Special Report No. 66 and the computed orbital elements are referred to the same mean coordinate system;

3. The Gaussian Method used to determine the orbit is an approximate method;

4. It is assumed that only two approximations are necessary in the differential process of computing the geographic latitude

and longitude.

I take this opportunity to express my gratitude to Dr. Ivan I. Mueller of the Department of Geodetic Science who has been my faculty adviser; to Mrs. Beatrice Miller of the Smithsonian Institution Astrophysical Observatory who advised me on the data published by the Observatory; to Dr. Nicholas T. Bobrovnikoff of the Department of Physics and Astronomy; to the United Stated Navy for providing the financial assistance and the opportunity to study Geodetic Science here at Ohio State; and to my fellow students whose thoughtful inquiries and interest in the subject served as a source of encouragement throughout the preparation of the thesis and the accompanying lengthy computations.

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1. INTRODUCTION

Since the early days of history man has shown a keen interest and an investigative concern about the earth, its characteristics, and its dimensions. Since the third century B.C. when Pythagoras stated that the earth was of spherical shape, many attempts have been made to measure its size and shape and to reduce the physical parameters and features on to maps and charts. Of the many existing earth sciences, geodesy is the one that primarily contains this field.

The inception of geodetic science can be traced back to Eratosthenes (circa 276 - 194 B.C.) who was the first man to measure the size of the earth. His method of measuring the length of a meridianal arc by timing the travel of a camel caravan between Alexandria and Syene (Aswan) and measuring the angle of incidence of the rays of the sun shining into a well at these places was crude but effective. The degree of accuracy he obtained was remarkable when we consider the tools he had available.

The science showed little development between this time and the seventeenth century. At this time the advent of the telescope, logarithmic tables and the method of triangulation (developed by the Dutch scientist Snellius in 1615) took place and geodesy was reborn.

The history of geodesy can be divided into periods based on the figure of the earth that was used as a reference during the respective periods. The studies and experiments of Pythagoras, Aristotle, and Eratosthenes brought the "flat", or plane era to an end and introduced the spherical era. Newton and Huygens, in the eighteenth century, introduced the ellipsoidal period which finally phased into the geoidal and telluroidal periods of this century.

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With the launching of the first artificial satellite and the beginning of the space age in this decade a new era in the field of geodesy is unfolding. By using the geoid as a reference surface we can determine the shape of the earth, but not its size. The orbiting artificial earth satellites will not replace the geoid or the telluroid but will provide the geodesist with a convenient reference station or "base-line in the sky" that can be used to measure both the size and shape of the earth.

Since the launching of our first artificial satellite (Explorer I) on 31 January 1958, the field of space science has expanded tremendously and with the successful results of the early and present projects the limits in this science are beyond imagination. At the present, the projects already underway and those planned for the immediate future suggest the following three areas of space activity: research, application, and exploration. [1, P.5]

The applications include meteorology, communications, and geodesy. No new principles, yet to be discovered are necessary; no major technological problems stand in our way.

In the field of meteorology, the satellites can provide, through telemetry, information on pressure, temperature, cloud coverage and storm patterns and also assist in predicting such mankillers as hurricanes, cyclones, etc. The day to day practical use of weather satellites is obvious.

In communications, satellite contributions are numerous: telephony, long-range radio communications, and international television.

This brings us to the third application of space activity: geodesy. Geodetic satellites can serve both research and application

interests. As to the latter, navigation and surveying come closest to the immediate interests of mankind. The geodesist can use the satellite in both a dynamic and a geometric approach. Dynamically, the perturbations in the orbits of the satellites caused by changes in the earth's gravitational attraction can be mathematically computed and analyzed and thus provide valuable information to assist in determining the gravity field of the earth. By observing the changes in the orbits of many satellites over a long period of time, information on regional gravity anomalies can be obtained. From these data we can compute the shape of our equipotential reference surface, the geoid. Further, observations of the nodal motion or the rate of change of the plane of the orbit of the satellite gives us information on the flattening of the earth. Therefore, in the dynamic approach we can determine the shape of the earth.

The size of the earth can be realized in the geometric approach. In this process the satellite will serve as a reference point or a triangulation station in the sky. Thus we can connect triangulation nets or stations that are widely separated by oceans and continents or are located in inaccessible areas where the conventional methods of triangulation would be awkward and cumbersome. If so desired, the existing geodetic systems could be reduced to a common world geodetic network or datum.

In this thesis a phase of the geometric use of the artificial earth satellite in geodesy will be discussed. In the geometric approach the geodesist can use the satellite in two basic ways. One method involves the observation of the satellite simultaneously from two or more stations. The other method requires that the orbit of the

satellite be precisely known. Then knowing the orbital parameters or characteristics and predicting their values for a desired time we observe the satellite from any number of stations on the earth.

The discussion of the problem will be treated in the following way:

First, the orbit of the satellite Explorer I (1958 Alpha) will be computed from the right ascension and declination of the satellite as observed at three different times from the tracking station at Olifants-fontein, South Africa. The three observations have been made at very short intervals during the same revolution of the satellite. The values of right ascension and declination are ~~apparent~~ MEAN ones ~~computed~~ REFERRED TO THE EPOCH 1950.0, REDUCED from camera films of Baker-Nunn tracking cameras. The geographic coordinates, latitude (ϕ) and longitude (λ) of the tracking station are known and its height (H) above sea level is given.

Second, the same satellite will be observed again from a station on the island of Curacao in the Netherland West Indies in the Caribbean. The geographic coordinates of this station will be assumed to be unknown. Then using the computed orbital parameters and observing the satellite's right ascension and declination from this station, the geographic co-ordinates of this unknown station will be found.

Third, the forward and back azimuths and the geodesic distance between the two stations will be computed.

This connection between the two points is called a tie and in this instance, an inter-continental tie. The distance between the stations is approximately 6,000 nautical miles. ~~The procedure discussed in the following chapters can be applied to any satellite observed from any location on the surface of the earth.~~

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2. DEFINITIONS AND ELEMENTS OF THE ORBIT

The motion of an artificial satellite in its orbit about the earth follows the same laws that apply to the movement of the planets in relation to the sun in the solar system.

Johannes Kepler (1571 - 1630), a German astronomer and mathematician, derived the laws of planetary motion from the observations of Tycho Brahe (1546 - 1601), Danish astronomer, with whom he worked as an assistant.

Kepler's Laws are:

I The orbit of each planet is an ellipse with the sun at one of its foci.

II The line (radius vector) joining the sun to each planet sweeps over equal areas of its ellipse in equal times.

III The squares of the orbital periods of the planets are proportional to the cubes of their mean distances from the sun (foci).

The relationship of an artificial satellite with respect to the earth is the same as that existing between the planets and the sun as stated in the above laws.

In other words, we can replace the term planet with satellite and replace the sun with the earth and the laws are still applicable. Now, since we know the basic form of the orbit is an ellipse we can easily describe the characteristics of the orbit. These orbital characteristics are known as parameters and can be considered to be constant quantities for a particular satellite. The first two parameters are properties of the ellipse itself and these are the semi-major axis, a, and the eccentricity, e. These two terms define any ellipse.

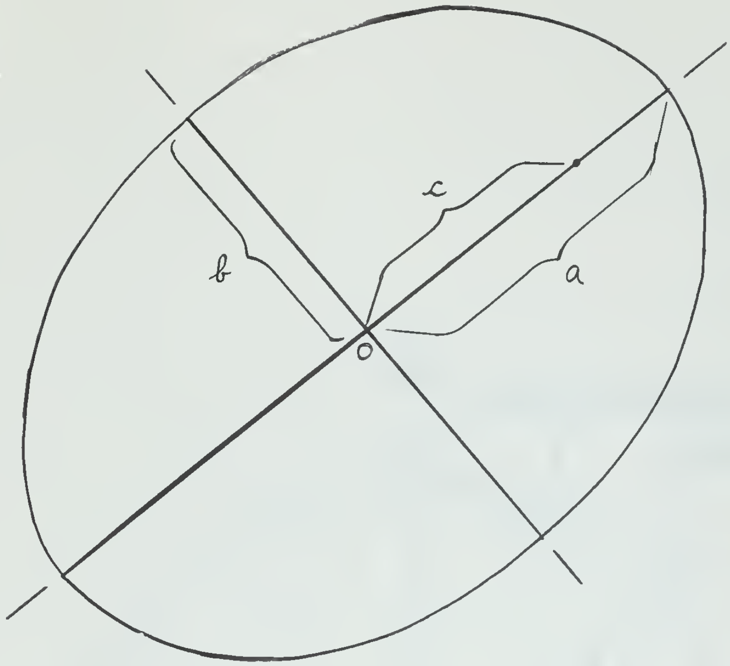


Figure 1.

Referring to Figure 1., the eccentricity is the ratio existing between the length of the semi-major axis and the distance of the foci, c, from the center, o, or:

$$e = c/a$$

The relationship $e^2 = (a^2 - b^2)/a^2$, where b is the minor semi-axis, also exists. The eccentricity of a circle is zero and that of a parabola is equal to one. The eccentricity of an ellipse is less than one but greater than zero. The eccentricity of the orbit of a satellite is nearly circular, that is, slightly greater than zero.

Two other parameters of the orbit are the angle of inclination, i, and the right ascension of the ascending node, Ω. These two values define the plane of the orbit in space. Referring to Figure 2, the angle i is measured from the plane of the equator to the plane of

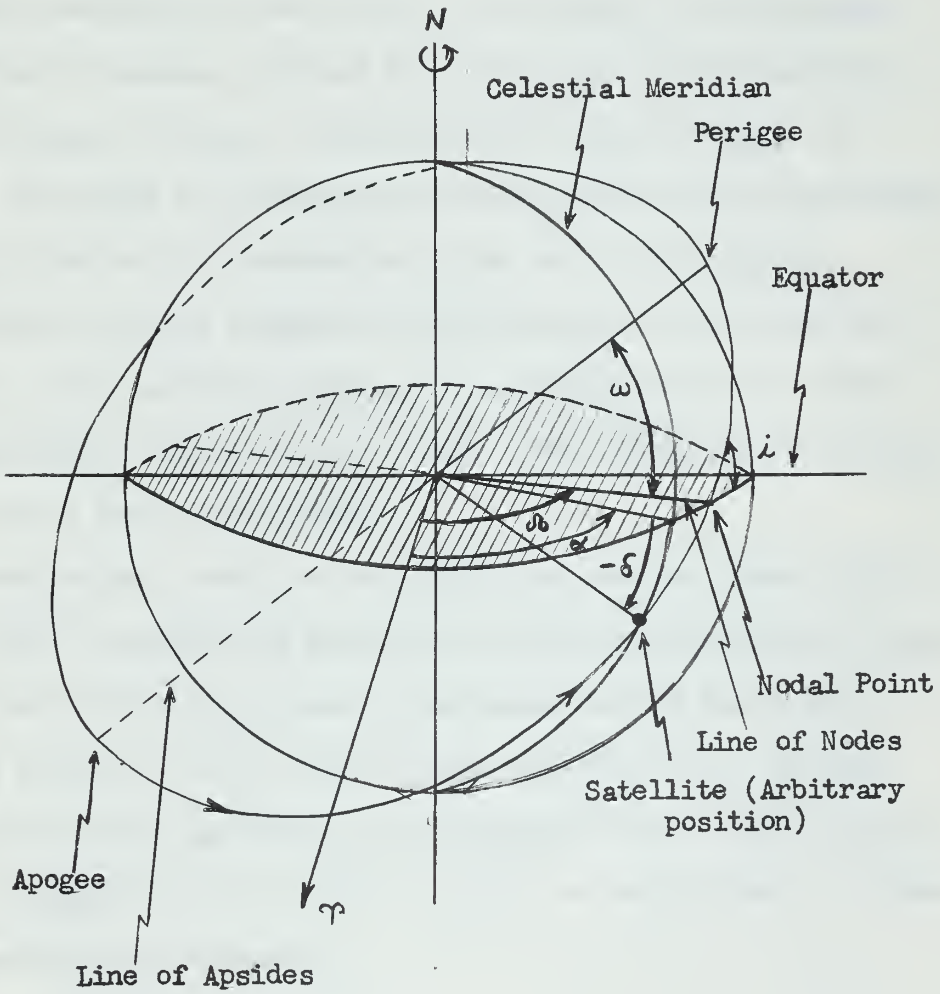

N
Celestial Meridian
Perigee
Equator
ω
i
☊
α
-δ
Nodal Point
Line of Nodes
Satellite (Arbitrary position)
Apogee
♈
Line of Apsides

FIGURE 2

the orbit. It can have a value between 0° and 180°. The angle Ω is measured eastward along the equatorial plane from the vernal equinox, to the ascending node. It can have any value between 0° and 360°. The ascending node is that point where the satellite crosses the equatorial plane from south to north.

Another parameter of the orbit is the argument of the perigee, w, or the angle measured from the equatorial plane at the ascending node to the major semi-axis (the extension of which contains the perigee). The angle w is measured in the direction of the satellite's motion and gives us the orientation of the orbit in its plane.

The sixth and last parameter is the time, T_o. Either the time of passage of the satellite through the ascending node or the time of passage through perigee may be chosen. The latter time is usually the one used in most computations.

In summarizing: the a and e specify the size and shape of the orbit; i and Ω specify the orientation of the orbital plane in space; w orients the orbit in its plane; T_o furnishes us with the value to locate the satellite in its orbital plane relative to the perigee.

To position the satellite at any specific time we must resort to a system of angular values called anomalies, of which there are three: mean M, eccentric E, and true V.

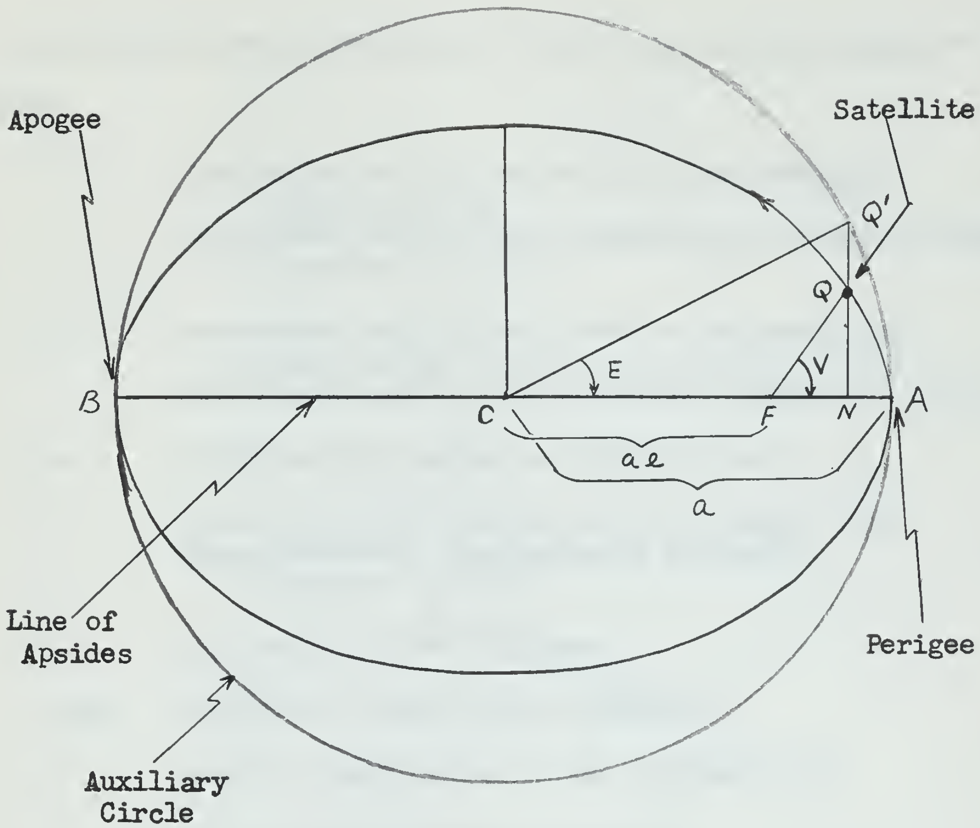


Figure 3.

Refer to Figure 3. Angle QFN is the true anomaly V, which is the angular distance of the satellite measured from the line of apsides along the plane of the orbit. Since the orbit of the satellite is an ellipse, the radius vector FQ, or the distance of the satellite from the foci, will not be constant. Therefore, the angular velocity is not constant. This must be so if Kepler's second law is to be fulfilled. If the angular velocity of the satellite was constant, the radius vector would sweep at an average rate and the eccentric anomaly would equal the mean anomaly. The mean anomaly is the theoretical angular velocity of a satellite moving uniformly through 360° of arc per orbital period.

The symbols and definitions of terms used in this paper are listed below:

a	Semi-major axis of the orbit of the satellite (Chapters 4, 5); Semi-major axis of the International Ellipsoid (earth) (Chapters 1, 2, 3)
b	Semi-minor axis of the orbit of the satellite (Chapters 4, 5); Semi-minor axis of the ellipsoid (Chapters 1, 2, 3)
c	Distance from center of orbit to foci
e	Eccentricity of orbital ellipse (Chapter 4,5); Eccentricity of International Ellipsoid (Chapters 1, 2, 3)
f	Flattening of the ellipsoid
f,g,h	Direction cosines (used together)
i	Angle of inclination of the orbital plane
k	Gaussian gravitational constant
n	Mean daily motion of satellite (velocity of the mean anomaly) in revolutions.
p	Auxiliary parameter of orbital ellipse (replaces a)
q	Geocentric distance of satellite at perigee (Chapter 5); Time interval: $(T_1 + T_3)k$ (Chapter 4)
r	Distance of satellite from observer
s	Geodesic distance
t	Time interval: $(T_3 - T_1)k$
u	Auxiliary parameter of orbit
x,y,z	Geocentric space rectangular coordinates
xa,ya,za	Approximate space rectangular coordinates of the observer
xo,yo,zo	Space rectangular coordinates of the observer.

A,B,C,D,E,F,G	Various terms used in Sodano's Fourth Method
E	Eccentric anomaly of satellite
G	Newton's Universal gravitational constant
H	Elevation above sea level
K, K1	Gaussian determinants
K_j	($j = 1,2,3,4,5,6$) Constants for International Ellipsoid used in Sodano's Fourth Method.
L	Difference in longitude between two points on the ellipsoid
M	Mean anomaly of satellite
N	Radius of curvature in the prime vertical
R	Geocentric distance of satellite
T_j	Time of observation ($j = 1, 2,3$)
T_o	Time of Epoch
V	True anomaly of satellite
X,Y,Z	Space rectangular coordinates of the satellite.
α	Flattening of the ellipsoid (Chapter 6)
α	Geocentric Right Ascension - the angular distance at the center of the earth measured eastward along the equator from the apparent vernal equinox to the celestial meridian passing through the satellite.
α'	Apparent Right Ascension - measured at the position of the observer.
α'_c	Apparent right ascension obtained through the approximate coordinates of observer.
α_{12}	Forward Azimuth
α_{21}	Back Azimuth
β	Parametric latitude (reduced latitude)
β_o	Maximum latitude of the geodesic.
γ	Normal gravity.

δ	Geocentric declination - the angular distance at the center of the earth measured northward or southward from the equator to the satellite. A satellite below the equator would have a south declination and the angle would have a minus value.
δ'	Apparent declination - measured at the observer's position.
δ'_c	Computed apparent declination obtained through the approximate coordinates of the observer.
θ	Spherical distance
θ_o	True spherical distance
θ_g	Greenwich hour angle of the apparent vernal equinox.
θ_ℓ	Local hour angle of the apparent vernal equinox.
λ	Geographic longitude (positive westward and negative eastward)
ρ	Geocentric distance of observer
σ	The angular motion of the satellite during the interval between observations. Usually expressed in radians.
ϕ	Geographic latitude (north positive and south negative)
ϕ'	Geocentric latitude
ω	Argument of the perigee. The angle measured eastward in the orbital plane from the line of nodes to the line of apsides.
Δ	Gaussian determinant
♈	Vernal Equinox - the point of intersection of the path of the sun with the equator at which the sun is passing from south latitude to north latitude.
Φ	Angle at satellite's position measured between the line to observer's position and the line to the earth's center (parallactic angle at middle observation).
Ψ	Angle at observer's position measured between the line to the earth's center and the line to position of the satellite at the time of the middle observation.

Ω Right ascension of the ascending node.

3. DETERMINATION OF THE MASS OF THE EARTH AND GAUSS' GRAVITATIONAL CONSTANT

The existing relationship between the earth and the satellite can be treated as a modified two body problem in which we assume that the two bodies are spheres and that their mass distribution is homogeneous. Under these circumstances, we can say that they will follow the laws of celestial mechanics and gravitation. That is, they will attract each other with a force that is proportional to the product of their masses and which varies inversely as the square of the distances between their centers. Since the mass of the satellite can be considered to be negligible compared to that of the earth and because of its proximity, it can be stated that the gravitational attraction of the earth is the only attracting force influencing the orbit. The force acting between the two bodies is:

$$F = k^2 \frac{m_e m_s}{r^2}$$

where r is the distance between the two bodies, m_e is the mass of the earth, m_s the mass of the satellite, and k equals the Gaussian gravitational constant. The numerical value of the gravitational constant depends on the units of mass, time, and distance chosen. [2, P.57]

Kepler's third law:

$$k^2 (m_e + m_s) = n^2 a^3$$

is a simple relation between mass, time, and distance that may conveniently be used in practical application of elliptic motion to find any one of these three quantities when the other two are known. The units of mass, time, and distance can be arbitrarily chosen and k can either be computed or derived from observations. In applying the units to determine k for use in observation of planetary orbits the following units are used: the sun's mass as the unit of mass, the

ephemeris day as the unit of time, and the semi-major axis of the earth's orbit (which approximates very closely, one astronomical unit) as the unit of distance. In the case of the artificial earth satellite a revision of these units must be made in order to find a realistic value of the Gaussian constant that will be applicable. Bear in mind that we are theorizing that the earth is the only body influencing the motion of the satellite. This is a sound theory but the other assumptions we made - that the earth is a sphere and that the mass distribution is homogeneous - are not absolutely correct. In fact, if the earth was spherical and the mass distribution was constant throughout, then the orbit of the satellite would be nearly perfect and very simple to predict. The plane of the orbit would be fixed, the parameters of the elliptical orbit would remain constant, and the center of the earth would act as one of the foci.

However, the earth is an ellipsoid of revolution with a bulge of mass at the equator and a non-homogeneous distribution of mass throughout. These characteristics plus the effect of the earth's atmosphere and other perturbations have a marked influence on the behavior of the satellite in its orbital travels.

The effect of the attraction of the equatorial mass of the earth is to pull the satellite toward the equator, thus decreasing the inclination. Behaving like a gyroscope the actual effect is to cause the plane of the orbit to precess westward for an eastward launch about the focus. This motion causes the perigee to advance to different points along the orbit. The perigee moves in the direction of the satellite motion when the inclination is less than $63.^{\circ}5$ and moves in the opposite direction when the inclination is greater than $63.^{\circ}5$. Consequently,

the elliptic orbit rotates about the focus in the orbital plane. Reductions of the periods of this perigee motion from the observations provide data to determine the shape of the earth. [11, P.4]

The non-homogeneity of the earth's mass distribution causes perturbations in the orbit of the satellite. By observing a number of satellites for long durations of time we can compute the deviations from the normal gravity and thus obtain values for the regional gravity anomalies.

Proceeding with the computation of the Gaussian constant, k, the following units will be used: the earth's mass as the unit of mass, the ephemeris day (86,400 seconds) as the unit of time, and the semi-major axis of the earth as the unit of distance. Compare these with the units mentioned earlier in this chapter that are used in conjunction with planetary orbits.

Note that all computations are based on the International Ellipsoid (Hayford, 1910) which has an equatorial radius of 6,378,388.000 meters and a flattening of 1/297.0. Here again rises an arbitrary value. There are many reference ellipsoids from which to choose but the variation in size from one to another is small. The International Ellipsoid has been adopted by many countries and it is the one that is used by Smithsonian Astrophysical Observatory in their computations, so I decided to use this as the reference body.

The international gravity formula adopted by the International Union of Geodesy and Geophysics in 1930 is [3, P.74] :

$$\gamma = \gamma_e (1 + \beta \sin^2 \phi + \epsilon \sin^2 2\phi)$$

where γ_e = gravity value at the equator, ϕ = latitude of station, β = gravitational flattening, ϵ = theoretical constant. The

coefficientts of the formula are:

$$\gamma = 978.049(1 + 0.0052884 \sin^2\phi - 0.0000059 \sin^2 2\phi) c\ m/sec^2$$

The gravity value at the equator is the resultant of the gravitational attraction and the centrifugal force due to the earth's rotation. Therefore,

$$T_e - w^2a = 978.0490 \text{ cm/sec}^2$$

where Te = attraction of the earth at the equator and w^2a = centrifugal force.

If $w^2a = (\frac{2\pi}{T})^2 r$, where w is the angular velocity of the earth, T is the sidereal day in seconds, and r the radius of the ellipsoid then:

$$(\frac{2\pi}{T})^2 r = 3.380545 \text{ cm/sec}^2$$

Then

$$T_e - 3.3805 = 978.0490$$

$$T_e = 978.0490 + 3.3805 = 981.4295 \text{ gal}$$

where a gal (from Galileo) equals a unit of acceleration equal to 1 cm/sec^2.

The mean density of the earth is computed from the formula [5]:

$$P_o = 2\pi G \sigma_m \frac{1 + e'^2}{e'^3} (\arctan e' - \frac{e'}{1 + e'^2})$$

where $P_o = T_e/a$ = attraction of earth at the equator divided by the semi-major axis, G = Newton's universal constant of gravitation which equals $6.673 \pm 0.003 \times 10^{-8}$ cgs, e' = the secondary eccentricity, and σ_m = the mean density of the earth. To solve the equation, two of the three quantities Te, a, or σ m must be known. In our example, the Te has been determined and the value of the semi-major axis of the International Ellipsoid is used. The computed mean density is (see Table 1):

$$\sigma_m = 5.51300437 \text{ gm/cm}^3$$

The volume of the International Ellipsoid is computed from the formula:

[illegible]

$$V = 4/3\ \pi\ a^2 b = 1.083320781 \times 10^{27}\ \text{cm}^3$$

where b = 6,356,911.9462 meters, the semi-minor axis of the ellipsoid.

Using these values, the computed mass of the earth is:

$$M = V\sigma = (1.083320781 \times 10^{27})\ (5.51300437) = 5.97235220 \times 10^{27}\ \text{gm.}$$

To obtain the proper unit for k in terms of mass of the earth for unit mass, time in days, and distance in terms of the earth's semi-major axis the following conversion is made:

$$\frac{1}{\text{sec}^2}\ \text{cm}^3 = \left[\frac{\text{cm}^3}{\text{gm sec}^2}\right] \text{gm} = G\,\frac{(86{,}400)^2\ (5.9723522 \times 10^{27})}{(6{,}378{,}38800)^3} = k$$

and $k = 107.0731376$ and $k^2 = 11464.6568$

TABLE 1

COMPUTATION OF VOLUME, DENSITY, AND MASS OF THE EARTH; COMPUTATION OF k

DIMENSIONS OF THE INTERNATIONAL ELLIPSOID

α = 1/297
a = 6.378,388.0000 meters
b = 6,356,911.9462 meters
e = 0.0819918898
e^2 = 0.0067226700
e' = 0.0822688896
e'^2 = 0.0067681702
e'^3 = 0.0005568098

$$\frac{1+e'^2}{e'^3} = \frac{1.0067681702}{.0005568098} = 1808.10066$$

$$\text{ARCTAN } e' = e' - \frac{e'^3}{3} + \frac{e'^5}{5} - \frac{e'^7}{7} + \cdots$$

$$\cdots = 0.0820840365$$

$$\frac{e'}{1+e'^2} = 0.0817158230$$

$$\left(\frac{2\pi}{T}\right)^2 r = \left(\frac{2 \times 3.1415926535}{86,400}\right)^2 6,378,38800. = 3.3805456400$$

$$V = 4/3\,\pi\, a^2 b = 1.083320781 \times 10^{27} \text{ cm}^3$$

$$\sigma_m = \frac{T_e}{2\pi G \dfrac{1+e'^2}{e'^3}\left[\arctan e' - \dfrac{e'}{1+e'^2}\right](a)} = 5.51300437 \text{ gm/cm}^3$$

$$M = V\sigma_m = 5.972352200 \times 10^{27} \text{ gm}$$

$$k = G\frac{1}{\sec^2}\text{ cm}^3 = G\left[\frac{\text{cm}^3}{\text{gm sec}^2}\right] \text{ gm} = G\frac{(86,400)^2(5,972352 \times 10^{27}}{(6,378,38800)^3} =$$

107.0731376

4. DETERMINATION OF THE ORBIT

Since the orbit of the satellite is defined by six parameters then six independent quantities must be given by the observations in order that the parameters may be determined. A single observation gives two quantities, the angular coordinates of the body. Therefore, three complete observations are sufficient to define the orbit. [4, P.191]

The apparent positions of the observed satellite are obtained by measuring its angular distances and directions from a background of stars on the film of the Baker-Nunn camera. The stars on the film can be identified from catalogues containing their right ascension and declination. With these coordinates known, then the right ascension and declination of the satellite can be computed. Catalogues containing the observed values of time, right ascension, and declination reduced from the camera film are published about six months after the observations were made. These reduced values are quite accurate. However, a more precise reduction of the films is made by the Smithsonian Institution Astrophysical Observatory Photoreduction Division. These results are published and made available about 18 months after the observations are taken.

In this paper the computation of the orbit of Explorer I (1958 Alpha) satellite will be reduced from observations of right ascension and declination from the Baker-Nunn Camera Tracking Station at Olifantsfontein, South Africa. Date of the observations was 18 December 1960. The observed data [7, p.10]:

Time (UT)	α'	δ'
$22^h\ 29^m\ 10^s.14$	$5^h\ 07^m\ 06^s$	$-47^\circ\ 22'00''$
$22^h\ 31^m\ 04^s.42$	$7^h\ 28^m\ 54^s$	$-51^\circ\ 06'00''$
$22^h\ 32^m\ 03^s.10$	$8^h\ 33^m\ 00^s$	$-49^\circ\ 25'00''$

These data are not those precisely reduced by the Photo-reduction

Division but the precision of the observations is as follows: [7, p.3]

Standard Error in timing (σ_t) $\quad .005^s < \sigma_t \leqq .02^s$
Standard Error in direction (σ_D) $\quad 2'.7 < \sigma_D \leqq 3'.5$

The geographical coordinates and height of the tracking station are [8, p.7]:

$$\phi = 25^\circ\ 57'\ 34''.70 \quad S$$
$$\lambda = 28^\circ\ 14'\ 51''.10 \quad E$$
$$H = 1544 \text{ meters}$$

There are two classical methods available for determining the orbit— one developed by Laplace in 1780 and the other by Gauss in 1801. Both methods were devised for use in determining the orbits of comets and planets about the sun but are also applicable for use with artificial earth satellites. Since the development of these two methods many papers have been written on the theory of orbital determination but very little that is theoretically important or radically different has been added to the methods of these two men. The Laplacian method is not as complicated as the one devised by Gauss but it has two weaknesses. It is inaccurate when the intervals between observations are short and it assumes that the observer is in orbit about the focus of the ellipse. The Gaussian method, even though it is an approximate method was chosen to determine the orbit. There are other more modern solutions available but these are not as accurate.

The first step in the calculations is the conversion of the Universal time of the observations to sidereal time. This is required in order to introduce the vernal equinox into the coordinate system. The sidereal time at any instant is the hour angle of the vernal equinox.

The origin of the coordinate system is at the center of mass of the ellipsoid. The x -axis points toward the vernal equinox, the y-axis

is directed toward a point 90° eastward from the vernal equinox, and the z-axis coincides with the north pole of rotation. The x-y plane lies in the plane of the equator.

The rectangular space coordinates of the observer are computed from the given latitude and longitude of the tracking station. Then, by changing the signs of these values the coordinates of the center of the earth are obtained. The observer's position is computed as follows:

$$x_j = (N+H)\cos\phi\cos\theta_{1_j} \qquad (j = 1, 2, 3)$$
$$y_j = (N+H)\cos\phi\sin\theta_{1_j}$$
$$z_j = (N+H)\sin\phi\,(1 - e^2)$$

The position of the observer, the earth's center, and eventually the position of the satellite are all converted to rectangular space coordinates to provide a more convenient and a common system of coordinates.

The next step is to compute the direction cosines from the observer's position. These are functions of the observed (apparent) right ascension and declination and are obtained from:

$$f_j = \cos\delta'_j\cos\alpha'_j \qquad (j = 1, 2, 3)$$
$$g_j = \cos\delta'_j\sin\alpha'_j$$
$$h_j = \sin\delta'_j$$

The orientation of these axes: f-axis points toward the vernal equinox, g-axis perpendicular to f-axis, and the h-axis parallel to the axis of rotation of the earth and perpendicular to the f- and g-axes. The direction cosines give us a line in space from the observer's position to the satellite. The computations of these three steps for the three observations are contained in Table 2.

TABLE 2

COMPUTATION OF SIDEREAL TIME, COORDINATES OF EARTH CENTER, DIRECTION COSINES

Date	18 December 1960	18 December 1960	18 Dec. 1960
Sidereal Time At 0^h UT	5-46-25.419	5-46-25.419	5-46-25.419
UT Time Interval	22-29-10.140	22-31-04.420	22-32-03.100
UT To Sidereal Time	3-41.633	3-41.946	3-42.107
Greenwich Sidereal Time	28-19-17.192	28-21-11.785	28-22-10.626
λ (Add East, Subtract West)	+1-52-59.407	+1-52-59.407	+1-52-59.407
Local Sidereal Time (19 Dec)	$6^h12^m16^s.599$	$6^h14^m11^s.192$	$6^h15^m10^s.033$
θ	93°04'08".985	93°32'47".880	93°47'30".495
$\sin\theta_1$	.99856564	.99808478	.99781094
$\cos\theta_1$	-.05354139	-.06186091	-.06613118
α'	5^h-07^m-06^s	7^h-28^m-54^s	8^h-33^m-00^s
α'	76°46'30"	112°13'30"	128°15'00"
$\sin\alpha'$	.97347917	.92570563	.78531693
$\cos\alpha'$	.22877565	-.37824474	-.61909395
δ'	47°22'00" S	51°06'00" S	49°25'00" S
$\sin\delta'$	-.73570317	-.77824315	-.75946058
$\cos\delta$	.67730410	.62796306	.65055333
ϕ	→	25°57'34".7 S	←
$\sin\phi$	→	-.43773790	←
$\sin^2\phi$	→	.19161447	←
$\cos\phi$	→	.89910263	←
a	→	6,378,388.0000	←
e^2	→	.00672267	←
$1 - e^2$	→	.99327733	←
$e^2\sin^2\phi$	→	.00128816	←
$1 - e^2\sin^2\phi$	→	.99871184	←
$(1-e^2\sin^2\phi)^{\frac{1}{2}}$	→	.99935571	←
$a/(1-e^2\sin^2\phi)^{\frac{1}{2}} = N$	→	6,382,500.18 meters	←
H	→	1544 meters	←
H/a	→	.00024207	←
N/a	→	1.00064471	←
N/a + H/a = N+H	→	1.00088678	←
$-(N+H)\cos\phi\cos\theta_\ell = x$	+.04818189	+.05566863	+.05951144
$-(N+H)\cos\phi\sin\theta_\ell = y$	-.89860916	-.89817643	-.89793000
$-(N+H)\sin\phi(1-e^2) = z$	+.43518070	+.43518070	+.43518070
$\cos\delta'\cos\alpha' = f$	+.15495069	-.23752372	-.40275363
$\cos\delta'\sin\alpha' = g$	+.65934143	+.58130894	+.51089054
$\sin\delta' = h$	-.73570317	-.77824315	-.75946058

The next step is to determine the distance of the satellite from the observer's position and from the center of the earth. This is done by solving the triangle formed at the time of the middle observation between the center of the earth, the observer, and the satellite. To solve the triangle the satellite is "stopped" in its orbit at the time of the second or middle observation, (T_2).

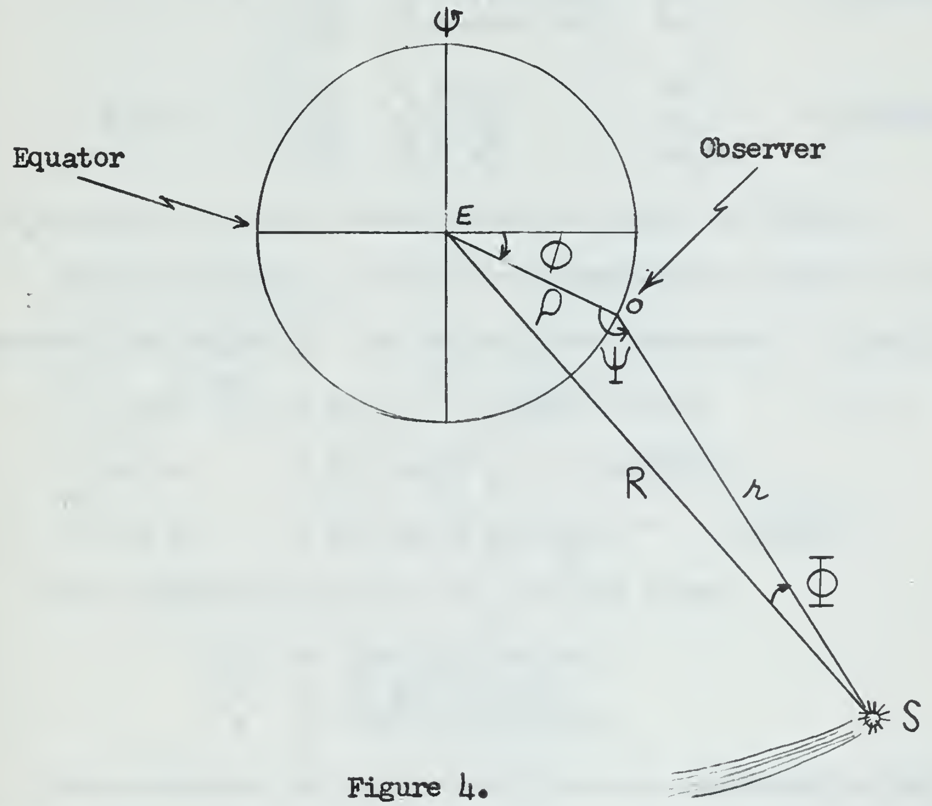


Figure 4.

The geometric situation is depicted in Figure 4. Since SOE form a triangle at T_2 the quantities r_2 and R_2 must satisfy the equation [4,p.201] :

$$R_2^2 = r^2 + \rho^2 - 2r\rho \cos\Psi$$

This equation treats the geometric situation. In the Gaussian method a formula relating the two unknown quantities r and R applies dynamically [4,p.235] :

$$r_2 = \frac{K}{\Delta} + \frac{t^2 K_1}{4\Delta R_2^3}$$

where K, Δ, and K_1 are determinants involving the coordinates of the earth's center and the direction cosines from the observer's position. These determinants and their values are:

$$\Delta = \begin{bmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{bmatrix} = -0.00204368$$

$$K = -\tfrac{1}{2} \begin{bmatrix} f_1 & x_1 + x_3 - 2x_2 & f_3 \\ g_1 & y_1 + y_3 - 2y_2 & g_3 \\ h_1 & z_1 + z_3 - 2z_2 & h_3 \end{bmatrix} = +0.00018897$$

$$K_1 = - \begin{bmatrix} f_1 & x_1 + x_3 & f_3 \\ g_1 & y_1 + y_3 & g_3 \\ h_1 & z_1 + z_3 & h_3 \end{bmatrix} = -0.45716431$$

The solutions of these determinants are shown in Table 3.

The two unknowns r_2 and R_2 are expressible in terms of a single unknown, the angle Φ, by means of the equations [4,p.235] :

$$\rho_2 \cos \Psi_2 = x_2 f_2 + y_2 g_2 + z_2 k_2 \qquad (0 < \Psi \leqq \pi)$$

$$N \sin m = \rho_2 \sin \Psi_2 = +.48507360$$

$$N \cos m = \rho_2 \cos \Psi_2 - \frac{K}{\Delta} = -.78155146$$

The computed values for Ψ_2, N, and m are:

$$\begin{aligned} \Psi_2 &= 150^\circ\ 58'\ 12''.064 \\ N &= \pm\, 0.91984731 \\ m &= 148^\circ\ 10'\ 26''.395 \end{aligned}$$

The solutions for these quantities are contained in Table 4. N, m, and M are intermediate parameters used in the solution for the angle Φ, where [4, p.212] :

$$M = \frac{4 \ \Delta \ \rho_2^3 \ N \sin^3 \Psi}{t^2 \ K_1} = +0.16344533$$

The sign of N is chosen so that M shall be positive.

The equation for the solution of the angle Φ is [4, p.213]:

$$\sin^4 \Phi = M \sin (\Phi + m)$$

which is a transcendental equation. The quantities M and m are known and M is positive. One solution of the problem results in the position of

TABLE 3

SOLUTION OF DETERMINANTS

Δ	$f_1\ g_2\ h_3$	=	$-.068\ 407\ 82$
	$f_2\ g_3\ h_1$	=	$.089\ 276\ 57$
	$f_3\ g_1\ h_2$	=	$.206\ 664\ 15$
	$f_3\ g_2\ h_1$	=	$.172\ 245\ 98$
	$f_1\ g_3\ h_2$	=	$-.061\ 607\ 94$
	$f_2\ g_1\ h_3$	=	$.118\ 938\ 54$
K	$f_1\ (y_1 + y_3 - 2y_2)h_3$	=	$.000\ 021\ 92$
	$(x_1 + x_3 - 2x_2)g_3h_1$	=	$.001\ 369\ 62$
	$f_3g_1(z_1 + z_3 - 2z_2)$	=	0
	$f_3\ (y_1 + y_3 - 2y_2)\ h_1$	=	$-.000\ 055\ 20$
	$f_1\ g_3(z_1 + z_3 - 2z_2)$	=	0
	$(x_1 + x_3 - 2x_2)\ g_1h_3$	=	$.001\ 824\ 68$
K_1	$f_1\ (y_1 + y_3)\ h_3$	=	$.211\ 414\ 83$
	$(x_1 + x_3)\ g_3\ h_1$	=	$-.040\ 478\ 02$
	$f_3\ g_1\ (z_1 + z_3)$	=	$-.231\ 126\ 34$
	$f_3\ (y_1 + y_3)\ h_1$	=	$-.532\ 327\ 35$
	$f_1\ g_3\ (z_1 + z_3)$	=	$.068\ 900\ 28$
	$(x_1 + x_3)\ g_1\ h_3$	=	$-.053\ 926\ 77$

TABLE 4

SOLUTION OF Ψ_2, N, m.

$x_2 f_2$	=	$-.013\ 222\ 62$	
$y_2 g_2$	=	$-.522\ 117\ 99$	
$z_2 h_2$	=	$-.338\ 676\ 40$	
$x_2 f_2 + y_2 g_2 + z_2 h_2$	=	$-.874\ 017\ 01$	$= \rho_2 \cos \Psi_2$
x_2^2	=	$.003\ 099\ 00$	
y_2^2	=	$.806\ 720\ 90$	
z_2^2	=	$.189\ 382\ 24$	
$x_2^2 + y_2^2 + z_2^2$	=	$.999\ 202\ 14$	
$(x_2^2 + y_2^2 + z_2^2)^{\frac{1}{2}}$	=	$.999\ 600\ 99$	$= \rho_2$
$x_2 f_2 + y_2 g_2 + z_2 h_2 / \rho_2$	=	$-\ .874\ 365\ 89$	$= \cos \Psi_2$
Ψ_2	=	$150^\circ\ 58'12''.064$	
$\sin \Psi_2$	=	$.485\ 267\ 23$	
$\rho_2 \sin \Psi_2$	=	$.485\ 073\ 60$	= N sin M
$\rho_2 \cos \Psi_2 - \frac{K}{\Delta}$	=	$-.781\ 551\ 46$	= N cos M
$N^2 \sin^2 m$	=	$.235\ 296\ 40$	
$N^2 \cos^2 m$	=	$.610\ 822\ 68$	
$N^2(\sin^2 m + \cos^2 m)$	=	$.846\ 119\ 08$	
N	=	$.919\ 847\ 31$	
$\rho_2 \sin \Psi_2 / N$	=	$.527\ 341\ 43$	= sin m
$\rho_2 \cos \Psi_2 - \frac{K}{\Delta} \Big/ N$	=	$-.849\ 653\ 47$	= cos m
m	=	$148^\circ\ 10'26''.395$	
Δ	=	$-.002\ 043\ 68$	
ρ_2^3	=	$.998\ 803\ 45$	
$\sin^3 \Psi$	=	$.114\ 272\ 81$	
N	=	$.919\ 847\ 31$	
t^2	=	$.011\ 485\ 88$	
K_1	=	$-.457\ 164\ 31$	
$4 \Delta \rho_2^3 N \sin^3 \Psi$	=	$-.000\ 858\ 24$	
$t^2 K_1$	=	$-.005\ 250\ 93$	
$4 \Delta \rho_2^3 N \sin^3 \Psi / t^2 K_1$	=	$.163\ 445\ 33$	= M

Solution for t: $t = \frac{k\,(T_3 - T_1)}{2}$

T_3	=	$22^h\ 32^m\ 03^s.10$	
T_1	=	$22^h\ 29^m\ 10^s.14$	
$T_3 - T_1$	=	$2^m\ 52^s.96$	
$T_3 - T_1$	=	.002 001 85 days	
k	=	107. 073 1376	
$k(T_3 - T_1)$	=	0.214 344 36	= 2t
t	=	0.107 172 18	

the observer and is not applicable. That is, if $r = 0$, $\rho = R$. The proper solution for the angle Φ must satisfy the inequality [4, p.213] :

$$\Phi < \pi - \Psi$$

In our problem, Ψ equals $150^\circ\ 58'12''.064$ so Φ must be less than $29^\circ 02'$. The solution may be obtained graphically be plotting the curves [4, p.213] :

$$y_1 = \sin^4 \Phi$$
$$y_2 = M \sin(\Phi + m)$$

Once the approximate value is found from the intersection of the two curves then the exact solution may be found by numerical trials or by a differential formula. The solution for the angle is lengthy and time - consuming. The approximate value from the intersection of the curves on the graph was found to be approximately 23°. The exact value, solved for by numerical trials was:

$$\Phi = 23^\circ\ 15'\ 49''.16$$

With this value known we can now solve for the two unknown sides of the triangle SOE in Figure 4, which will yield the distance of the satellite from the observer's position and the distance of the satellite from the earth's center for the second observation. The formulae used and the results are [4, p. 255] :

$$R_2 = \frac{\rho_2 \sin \Psi_2}{\sin \Phi} = 1.228\ 150\ 03$$

$$r_2 = \frac{\rho_2 \sin(\Psi + \Phi)}{\sin \Phi} = 0.254\ 280\ 87$$

The observer - satellite distances for the first and third observations are given by [4, p. 255]:

$$\begin{bmatrix} f_1 f_3 \\ g_1 g_3 \end{bmatrix} r_1 = \begin{bmatrix} x_1 f_3 \\ y_1 g_3 \end{bmatrix} - \frac{(1,3)}{(2,3)} \begin{bmatrix} x_2 f_3 \\ y_2 g_3 \end{bmatrix} + \frac{(1,2)}{(2,3)} \begin{bmatrix} x_3 f_3 \\ y_3 g_3 \end{bmatrix} + r_2 \frac{(1,3)}{(2,3)} \begin{bmatrix} f_2 f_3 \\ g_2 g_3 \end{bmatrix}$$

$$\begin{bmatrix} f_1 f_3 \\ g_1 g_3 \end{bmatrix} r_3 = \frac{(2,3)}{(1,2)} \begin{bmatrix} f_1 x_1 \\ g_1 y_1 \end{bmatrix} - \frac{(1,3)}{(1,2)} \begin{bmatrix} f_1 x_2 \\ g_1 y_2 \end{bmatrix} + \begin{bmatrix} f_1 x_3 \\ g_1 y_3 \end{bmatrix} + r_2 \frac{(1,3)}{(1,2)} \begin{bmatrix} f_1 f_2 \\ g_1 g_2 \end{bmatrix}$$

where

$$\frac{(1,3)}{(2,3)} = \frac{1}{\frac{1}{2} + \frac{q}{2t} + \frac{t^2}{4R_2^3}} = 2.934\ 11079$$

$$\frac{(1,2)}{(2,3)} = \frac{1 - \frac{q}{t} + \frac{t^2}{2R_2^3}}{1 + \frac{q}{t} + \frac{t^2}{2R_2^3}} = 1.943\ 20695$$

$$\frac{(2,3)}{(1,2)} = \frac{1 + \frac{q}{t} + \frac{t^2}{2R_2^3}}{1 - \frac{q}{t} + \frac{t^2}{2R_2^3}} = 0.51461323$$

$$\frac{(1,3)}{(1,2)} = \frac{1}{\frac{1}{2} - \frac{q}{2t} + \frac{t^2}{4R_2^3}} = 1.50993233$$

where

$$t = \frac{k\ (T_3 - T_1)}{2} = 0.10717218$$

$$q = \frac{k\ (T_1 + T_3)}{2} = -0.03445185$$

The time of the second observation, T_2, is the origin of time. Upon substituting these values into the above formulae and solving the determinants, the observer - satellite distances for the first and third observations are found to be:

$$r_1 = 0.23531756$$

$$r_3 = 0.27239982$$

The detailed solutions for R_2, r_1, r_2, r_3 and q are listed in Table 5.

The next step is to determine the geocentric space coordinates of the satellite for the time of each observation as follows [4, p.256]:

r	0.23531756	0.25428087	0.27239982
$X = r\ f - x$	-0.01171927	-0.11606637	-0.16922145
$Y = r\ g - y$	+1.05376378	+1.04599217	+1.03709649

TABLE 5

DETERMINATION OF R_2, r_2, q, r_1, r_3.

$\rho_2 \sin \Psi_2$	=	0.485 073 60		
$\sin \Phi$	=	0.394 962 82		
$\rho_2 \sin \Psi_2 / \sin \Phi$	=	1.228 150 03	=	R_2
Ψ_2	=	150° 58'12".06		
	=	23° 15'49".16		
$(\Psi_2 + \Phi)$	=	174° 14'01".22		
$\sin(\Psi_2 + \Phi)$	=	0.10047158		
$\rho_2 \sin(\Psi_2 + \Phi)$	=	0.10043149		
$\rho_2 \sin(\Psi_2 + \Phi)/\sin \Phi$	=	.25428087	=	r_2

T_3	= $22^h\ 32^m\ 03^s.10$	T_2	= $22^h\ 31^m\ 04^s.42$
T_2	= $22^h\ 31^m\ 04^s.42$	T_1	= $22^h\ 29^m\ 10^s.14$
$T_3 - T_2$	= (+) $58^s.68$	$T_2 - T_1$	= (−) $1^m\ 54^s.28$

$T_2 - T_1$	= (−)	$1^m\ 54^s.28$
$T_3 - T_2$	= (+)	$58^s.68$
$T_1 + T_3$	= (−)	$55^s.60$
$T_1 + T_3$ (days)	=	−0.00064352
k	=	107.0731376
$k\,(T_1 + T_3)$	=	−0.06890371
$k\,(T_1 + T_3)/2$	=	−0.03445185 = q

t	0.10717218	$4R_2^3$	7.40993264
2t	0.21434436	q	−0.03445185
t^2	0.01148588	q/2t	−0.16073131
R_2	1.22815003	q/t	−0.32146262
R_2^3	1.85248316	$t^2/2R_2^3$	0.00310013
$2R_2^3$	3.70496632	$t^2/4R_2^3$	0.00155007

$1 - q/t + t^2/2R_2^3 =$ 1.32456275 $\quad 1 + q/t + t^2/2R_2^3 =$ 0.68163751

$$1\Big/\tfrac{1}{2} + \frac{q}{2t} + \frac{t^2}{4R_2^3} = \frac{(1,3)}{(2,3)} = 2.93411079$$

$$1 - \frac{q}{t} + \frac{t^2}{2R_2^3}\Big/ 1 + \frac{q}{t} + \frac{t^2}{2R_2^3} = \frac{(1,2)}{(2,3)} = 1.94320695$$

$$1 + \frac{q}{t} + \frac{t^2}{2R_2^3}\Big/ 1 - \frac{q}{t} + \frac{t^2}{2R_2^3} = \frac{(2,3)}{(1,2)} = 0.51461323$$

$$1\Big/\tfrac{1}{2} - \frac{q}{2t} + \frac{t^2}{4R_2^3} = \frac{(1,3)}{(1,2)} = 1.50993223$$

$$\begin{array}{rcr} x_1 g_3 & & 0.02461567 \\ y_1 f_3 & & 0.36191810 \\ \hline x_1 g_3 - y_1 f_3 = \begin{bmatrix} x_1 f_3 \\ y_1 g_3 \end{bmatrix} & = & -\ 0.33730243 \end{array}$$

$$\begin{array}{rcr} x_2 g_3 & & 0.02844058 \\ y_2 f_3 & & 0.36174382 \\ \hline x_2 g_3 - y_2 f_3 = \begin{bmatrix} x_2 f_3 \\ y_2 g_3 \end{bmatrix} & = & -\ 0.33330324 \end{array}$$

$$\frac{(1,3)}{(2,3)} \begin{bmatrix} x_2 f_3 \\ y_2 g_3 \end{bmatrix} = -\ 0.97794863$$

$$\begin{array}{rcr} x_3 g_3 & & 0.03040383 \\ y_3 f_3 & & 0.36164457 \\ \hline x_3 g_3 - y_3 f_3 = \begin{bmatrix} x_3 f_3 \\ y_3 g_3 \end{bmatrix} & = & -\ 0.33124074 \end{array}$$

$$\frac{(1,2)}{(2,3)} \begin{bmatrix} x_3 f_3 \\ y_3 g_3 \end{bmatrix} = -\ 0.64366931$$

$$\begin{array}{rcr} f_2 g_3 & & -\ 0.12134862 \\ g_2 f_3 & & -\ 0.23412429 \\ \hline f_2 g_3 - g_2 f_3 = \begin{bmatrix} f_2 f_3 \\ g_2 f_3 \end{bmatrix} & = & 0.11277567 \end{array}$$

$$r_2 \frac{(1,3)}{(2,3)} \begin{bmatrix} f_2 f_3 \\ g_2 g_3 \end{bmatrix} = 0.08414060$$

$$\begin{array}{rcr} f_1 g_3 & & 0.07916284 \\ g_1 f_3 & & -\ 0.26555215 \\ \hline f_1 g_3 - g_1 f_3 = \begin{bmatrix} f_1 f_3 \\ g_1 g_3 \end{bmatrix} & = & 0.34471499 \end{array}$$

$$\begin{bmatrix} f_1 f_3 \\ g_1 g_3 \end{bmatrix} r_1 = 0.08111749$$

$$r_1 = .08111749/.34471499 = 0.23531756$$

$$\begin{array}{rcr} f_1 y_1 & & -\ 0.13924011 \\ g_1 x_1 & & 0.03176832 \\ \hline f_1 y_1 - g_1 x_1 = \begin{bmatrix} f_1 x_1 \\ g_1 y_1 \end{bmatrix} & = & 0.17100843 \end{array}$$

$$\frac{(2,3)}{(1,2)} \begin{bmatrix} f_1 x_1 \\ g_1 y_1 \end{bmatrix} = -\ 0.08800320$$

$$\begin{array}{rcr} f_1 y_2 & & -\ 0.13917306 \\ g_1 x_2 & & 0.03670463 \\ \hline f_1 y_2 - g_1 x_2 = \begin{bmatrix} f_1 x_2 \\ g_1 y_2 \end{bmatrix} & = & -\ 0.17587769 \end{array}$$

$$\frac{(1,3)}{(1,2)} \begin{bmatrix} f_1 x_2 \\ g_1 y_2 \end{bmatrix} = -\ 0.26556339$$

$$\begin{array}{llll} & f_1y_3 & & -\ 0.13913487 \\ & g_1x_3 & & \underline{\ \ 0.03923836} \\ f_1y_3 - g_1x_3 & = \begin{bmatrix} f_1x_3 \\ g_1y_3 \end{bmatrix} & = & -\ 0.17837323 \end{array}$$

$$\begin{array}{llll} & f_1g_2 & & \ \ 0.09007422 \\ & g_1f_2 & & \underline{-\ 0.15660923} \\ f_1g_2 - g_1f_2 & = \begin{bmatrix} f_1f_2 \\ g_1g_2 \end{bmatrix} & = & \ \ 0.24668345 \end{array}$$

$$r_2 \ \frac{(1,3)}{(1,2)} \begin{bmatrix} f_1f_2 \\ g_1g_2 \end{bmatrix} = 0.09471334$$

$$\begin{bmatrix} f_1f_3 \\ g_1g_3 \end{bmatrix} r_3 = 0.09390030$$

$$r_3 = .09390030/.34471499 = 0.27239982$$

$Z = rh - z$	-0.60830457	-0.63307305	-0.64205763
$R = \sqrt{(x^2+y^2+z^2)}$	1.21679493	1.22815003	1.23143940

There is a check for accuracy of the computations up to this point in this last step. Note the agreement in the value for R_2 computed from the space coordinates and that determined from the formula, $(\rho_2 \sin \Psi_2)/\sin \Phi$, in Table 5.

The inclination of the orbital plane and the intersection of the orbital plane with the plane of the equator (XY plane) can now be solved using the geocentric space coordinates in a ratio of determinants. This solution will produce the first two parameters of the orbit that we are seeking: i and Ω [4, p.256] :

$$A : B : C = \begin{bmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{bmatrix} : \begin{bmatrix} Z_1 X_1 \\ Z_2 X_2 \end{bmatrix} : \begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{bmatrix}$$

See Table 6 for the evaluation of the ratio of these determinants and the detailed solutions for i and Ω.

Then,

$$\cos i = \frac{C}{\sqrt{A^2 + B^2 + C^2}} \qquad i = 32^\circ\ 34'20''.245$$

$$\sin \Omega \sin i = \frac{A}{\sqrt{A^2 + B^2 + C^2}}$$

$$\cos \Omega \sin i = \frac{-B}{\sqrt{A^2 + B^2 + C^2}} \qquad \Omega = 206^\circ\ 00'27''.603$$

If the inclination was greater than 90° the sign of the previous two equations would be reversed. The values i and Ω uniquely determine the position of the orbital plane in relation to the equatorial plane of the earth [4, p.146].

The next step is to find the auxiliary parameter u for each observation. Solution of this value is necessary in order to find the "longitude"

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

TABLE 6

EVALUATION OF DETERMINANTS A: B: C ; SOLUTION FOR i AND Ω .

$$Y_1Z_2 = -0.66710945$$
$$Y_2Z_1 = -0.63628182$$
$$Y_1Z_2 - Y_2Z_1 = \begin{bmatrix} Y_1Z_1 \\ Y_2Z_2 \end{bmatrix} = -0.03082763 = A$$

$$Z_1X_2 = 0.07060370$$
$$Z_2X_1 = 0.00741915$$
$$Z_1X_2 - Z_2X_1 = \begin{bmatrix} Z_1X_1 \\ Z_2X_2 \end{bmatrix} = 0.06318455 = B$$

$$X_1Y_2 = -0.01225826$$
$$X_2Y_1 = -0.12230654$$
$$X_1Y_2 - X_2Y_1 = \begin{bmatrix} X_1Y_1 \\ X_2Y_2 \end{bmatrix} = 0.11004828 = C$$

$$\sqrt{A^2+B^2+C^2} = 0.13058811$$

$$C/\sqrt{A^2+B^2+C^2} = 0.84271286 = \cos i$$

$$i = 32^\circ 34' 20''.245$$
$$\sin i = 0.53836329$$
$$A/\sqrt{A^2+B^2+C^2} = -0.23606766$$

$$\frac{A}{\sqrt{A^2+B^2+C^2}} \Big/ \sin i = -0.43849138 = \sin \Omega$$

$$B/\sqrt{A^2+B^2+C^2} = 0.48384612$$

$$-\frac{B}{\sqrt{A^2+B^2+C^2}} \Big/ \sin i = -0.89873535 = \cos \Omega$$

$$\Omega = 206^\circ 00' 27''.603$$

of the satellite from the position of nodal passage. The longitude of the satellite from the node is called the argument of the latitude.

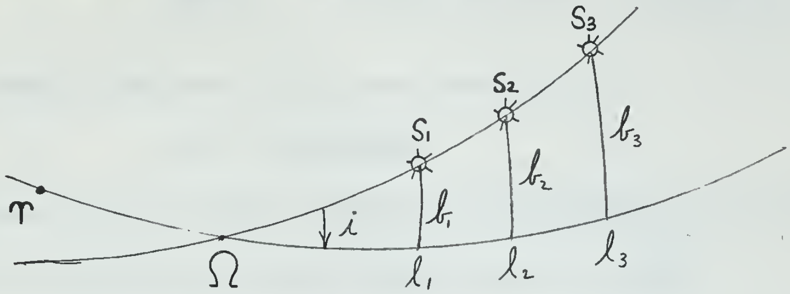


Figure 5.

Refer to Figure 5. The formulae that define u_1, u_2, u_3 are [4,p.238]:

$$\cos u_j = \cos b_j \cos (l_j - \Omega) \qquad (j = 1, 2, 3)$$

$$\sin u_j \cos_i = \cos b_j \sin (l_j - \Omega)$$

$$\sin u_j = \sin b_j / \sin_i$$

To determine the parameters a, e, and w, we must solve for the ratios of the triangles formed at the time of each observation. With the geocentric coordinates and the range of the satellite from the earth's center known the above equations may be substituted with the following expressions [4, p.253,257]:

$$\sin i \sin u_j = \frac{Z_j}{R_j} \qquad j = (1, 2, 3)$$

$$\cos i \sin u_j = \frac{Y_j}{R_j} \cos \Omega - \frac{X_j}{R_j} \sin \Omega$$

$$\cos u_j = \frac{Y_j}{R_j} \sin \Omega + \frac{X_j}{R_j} \cos \Omega$$

Then for abbreviation:

$$\sigma_1 = u_3 - u_2 = 2^{\circ}.44197583 = 0.04262050 \text{ radians}$$

$$\sigma_2 = u_3 - u_1 = 7^{\circ}.45516083 = 0.13011710 \text{ radians}$$

$$\sigma_3 = u_2 - u_1 = 5^{\circ}.01318500 = 0.08749658 \text{ radians}$$

Then the ellipse parameter p is determined and used in place of the semi-major axis, a.

[illegible]

[illegible]
[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]
[illegible]
[illegible]
[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]
[illegible]

The equation that defines p is [4, p.257] :

$$k\sqrt{p}\ (T_3-T_1) = \frac{R_2^2\ \sigma_2^3}{6\ \sigma_1\ \sigma_3} + \frac{R_1^2\ \sigma_2(2\sigma_3-\sigma_1)}{6\ \sigma_3} + \frac{R_3^2\sigma_2(2\sigma_1-\sigma_3)}{6\ \sigma_1}$$

Using this formula the value for p was found to be:

$$p = 0.83055354$$

With the values of p and u now known the eccentricity, e, and the argument of the perigee, w, can be found from [4, p.257]:

$$e \sin (u1 - w) = \frac{R_3(p-R_1) \cos \sigma_2 - R_1\ (p-R_3)}{R_1\ R_3 \sin\ \sigma_2}$$

$$e \cos (u_1-w) = \frac{p - R_1}{R_1}$$

Then the semi-major axis of the orbital ellipse is defined by

$$a = \frac{p}{(1 - e^2)}$$

The computed values for these three orbital elements are:

$$e = 0.32815828$$
$$a = 0.93078812$$
$$w = 82^\circ 54' 42''.337$$

The detailed computation of these parameters and the auxiliary parameters u_1, u_2, u_3, and p are listed in Tables 7, 8, and 9.

The next and final orbital parameter to be computed is the time of epoch, T_o (time of perigee passage). This value is computed from Kepler's equation:

$$M = E - e \sin E$$

after the true and eccentric anomalies are determined for the three observations. Values for To are:

	Observation #1	Observation #2	Observation #3
December	18.91454942	18.91453182	18.91455056

Detailed computations of T_o are listed in Table 10.

To verify the results of the computations, the orbital parameters were determined again using data published in Smithsonian Institution

Astrophysical Observatory Special Report #78 dated 25 October 1961. This publication contains both the mean elements and the smooth elements of Explorer I (1958 Alpha) satellite based on 82 observations made from the various Baker-Nunn Tracking Stations for the period 16 December through 29 December 1960. The published smooth elements are:

T_o = 37291.0 Modified Julian Days (Smithsonian Days)

$$w = (140^\circ.6 \pm 2) + 7^\circ.275\ (t-T_o) + 0^\circ.000427(t-T_o)^2 + 0^\circ.224 \cos w$$

$$\Omega = (186^\circ.278 \pm 9) - 4^\circ.896(t-T_o) - 0^\circ.000298(t-T_o)^2 + 0^\circ.00887 \cos w$$

$$i = (33^\circ.210 \pm 3) - 0^\circ.000554(t-T_o) + 0^\circ.50\times10^{-5}(t-T_o)^2 - 0^\circ.00473 \sin w$$

$$e = (.09971 \pm 8) - 0.000202(t-T_o) + 0.30 \times10^{-7}(t-T_o)^2 + 0.00045519 \sin w$$

$$M = (.94016 \pm 8)\ (13.435134 \pm 5)\ (t-T_o) + (0.0002839 \pm 6)\ (t-T_o)^2 - (.11 \pm 1) \times10^{-5}(t-T_o)^3 - (.000631 \times10^{-3}) \cos w$$

Standard error of one observation: $m = \pm 5'.43$.

The angles are expressed in degrees, e in terms of the earth's semi-major axis (6,378,388 m.), and M in revolutions. The Smithsonian Day for December 0, 1960 is 37268.0. Using the time of epoch, T_o = December 18.91453182 from the middle observation of our calculations and redesignating it as t, the term $(t - T_o)$ will have the following value:

Date	Smithsonian Day
December 0, 1960	37268.0
Date of Observation	+18.91453182
Date of Observation (t)	37286.91453182
Epoch (T_o)	37291.0
($t - T_o$)	-4.08546818 days

Detailed computations of the elements are contained in Table 11. The computed and published parameters are:

	Published	Computed
i	33°13'00.402	32°34'20.245
Ω	206°16'43".067	206°00'27".603
e	0.10011028	0.32815828
a	1.17098895	0.93078812
w	110°57'53".754	82°54'42".337

TABLE 7

COMPUTATION OF THE AUXILIARY PARAMETER, u

	Observation 1	Observation 2	Observation 3
Z_j/R_j	-0.49992366	-0.51546882	-0.52138792
$\sin i$	0.53836329	0.53836329	0.53836329
$(Z_j/R_j)/\sin i = \sin u$	-0.92859909	-0.95747394	-0.96846856
Y_j/R_j	0.86601592	0.85168110	0.84218232
X_j/R_j	-0.00963126	-0.09450504	-0.13741760
$\sin \Omega$	$\longrightarrow$	-0.43849138	$\longleftarrow$
$\cos \Omega$	$\longrightarrow$	-0.89873535	$\longleftarrow$
$Y/R \sin \Omega$	-0.37974052	-0.37345482	-0.36928969
$X/R \cos \Omega$	0.00865595	0.08493502	0.12350205
$Y/R\sin \Omega + X/R \cos \Omega = \cos u$	-0.37108457	-0.28851980	-0.24578764
u	248°13'02".845	253°13'50".311	255°40'21".424
$u_3 - u_2 = \sigma_1$	2°26'31".113		
σ_1 (radians)	0.04262050		
$u_3 - u_1 = \sigma_2$		7°27'18".579	
σ_2 (radians)		0.13011710	
$u_2 - u_1 = \sigma_3$			5°00'47".466
σ_3 (radians)			0.08749658

TABLE 8

DETERMINATION OF AUXILIARY PARAMETER, p.

$$k\sqrt{p}\,(T_3-T_1) = \frac{R_2^2\,\sigma_2^3}{6\sigma_1\sigma_3} + \frac{R_1^2\,\sigma_2(2\sigma_3-\sigma_1)}{6\sigma_3} + \frac{R_3^2\,\sigma_2(2\sigma_1-\sigma_3)}{6\sigma_1}$$

R_1^2	1.48058990	
R_2^2	1.50835250	
R_3^2	1.51644300	
σ_2^3	0.00220294	
$R_2^2\,\sigma_2^3$	0.00332281	
$6(\sigma_1\,\sigma_3)$	0.02237490	
$R_2^2\,\sigma_2^3/(6\,\sigma_1\,\sigma_3)$	0.14850614	*
$R_1^2\,\sigma_2$	0.19265006	
$(2\,\sigma_3-\sigma_1)$	0.13237266	
$6\,\sigma_3$	0.52497948	
$R_1^2\,\sigma_2(2\,\sigma_3-\sigma_1)/6\,\sigma_3$	0.04857637	*
$R_3^2\,\sigma_2$	0.19731517	
$(2\,\sigma_1-\sigma_3)$	-0.00225558	
$6\,\sigma_1$	0.25572300	
$R_3^2\,\sigma_2(2\,\sigma_1-\sigma_3)/6\,\sigma_1$	-0.00174040	*
$k\sqrt{p}\,(T_3-T_1) = *+*+* =$	0.19534211	
$K\,(T_3-T_1)$	0.21434436	
$*+*+*/k(T_3-T_1) = \sqrt{p} =$	0.91134710	
p	0.83055354	

TABLE 9

COMPUTATION OF THE ORBITAL PARAMETERS e, a, w.

$$e \sin (u_1 - w) = \frac{R_3 (p-R_1)\cos \sigma_2 - R_1(p-R_3)}{R_1 R_3 \sin \sigma_2}$$

R_3	1.23143940
$(p-R_1)$	-0.38624139
$\cos \sigma_2$	0.99154671
R_1	1.21679493
$(p-R_3)$	-0.40088586
$R_3(p-R_1)$	-0.47563287
$R_3(p-R_1)\cos \sigma_2$	-0.47161221
$R_1(p-R_3)$	-0.48779588
$R_3(p-R_1)\cos \sigma_2 - R_1(p-R_3)$	0.01618367
$\sin \sigma_2$	0.12975025
$R_1R_3 \sin \sigma_2$	0.19441897
$e \sin (u_1 - w)$	0.08324121

$e \cos (u_1 - w) = (p-R_1)/R_1 = -0.31742521$

$e^2\sin^2(u_1-w)$	0.00692910
$e^2\cos^2(u_1-w)$	0.10075876
$e^2\left[\sin^2(u_1-w)+\cos^2(u_1-w)\right]$	0.10768786
e	0.32815828

$p/1-e^2 = a$ 0.93078812

$e \sin (u_1-w)$	0.08324121
$\sin (u_1-w)$	0.25366177
$e \cos (u_1-w)$	-0.31742520
$\cos (u_1-w)$	-0.96729298
(u_1-w)	165°18'20".508
u_1	248°13'02".845
w	82°54'42".337

TABLE 10

COMPUTATION OF TIME OF EPOCH, T_o

	Observation #1	Observation #2	Observation #3
$V = u-w$	165°18'20".508	170°19'07".974	172°45'39".087
$V/2$	83°39'10".254	85°09'33".987	86°22'49".543

$$\tan E/2 = \sqrt{(1-e)/(1+e)}\ \tan V/2$$

	Observation #1	Observation #2	Observation #3
$1-e$	→	0.67184172	←
$1+e$	→	1.32815828	←
$(1-e)/(1+e)$	→	0.50584462	←
$\sqrt{(1-e)/(1+e)}$	→	0.71122754	←
$\tan V/2$	7.7555753	11.808428	15.808374
$\tan E/2$	5.51597874	8.39847920	11.24335095
$E/2$	79°43'27".723	83°12'35".307	84°55'02".654
E	159°26'55".446	166°25'10".614	169°50'05".308

$$M = E - e \sin E$$

	Observation #1	Observation #2	Observation #3
E (radians)	2.78290541	2.90457022	2.96417658
$\sin E$	0.35104533	0.23480935	0.17648680
$e \sin E$	0.11519843	0.07705463	0.05791560
M (radians)	2.66770698	2.82751559	2.90626098
M	153°25'16".711	162°00'16".966	166°30'59".358

$$n = k/\sqrt{a^3}$$

	Observation #1	Observation #2	Observation #3
k	→	107.0731376	←
a^3	→	0.80640366	←
$\sqrt{a^3}$	→	0.89799981	←
n	→	119.2351450 rad./day	←
$E-e \sin E/n = t - T_o$	0.02237350	0.02371378	0.02437420
t_j (days) $j = (1,2,3)$	Dec 18.93692292	Dec 18.93824560	Dec 18.938924768
T_o	Dec 18.91454942	Dec 18.91453182	Dec 18.91455056

TABLE 11

ORBITAL ELEMENTS DETERMINED FROM PUBLISHED DATA

$(t-T_o) = -4.08546818$ days

$(t-T_o)^2 = 16.69105025$

$(t-T_o)^3 = -68.19075469$

$w = 110^\circ\ 48'\ 40''.896$
$\sin w = 0.93475525$
$\cos w = -0.35529231$

$$w = (140^\circ.6) + (7^\circ.275)(-4.08546818) + (^\circ.000427)(16.69105025) - (^\circ.224)(-.35529231)$$

$$w = 110^\circ\ 57'\ 53''.754$$

$$\Omega = (186^\circ.278) - (4^\circ.896)(-4.08546818) - (^\circ.000298)(16.69105025) - (^\circ.00887)(-.35529281)$$

$$\Omega = 206^\circ\ 16'\ 43''.067$$

$$i = (33^\circ.210) - (^\circ.000554)(-4.08546818) + (^\circ.000005)(16.69105025) + (^\circ.00473)(.93475525)$$

$$i = 33^\circ\ 13'\ 00''.402$$

$$e = (.09971) - (.000202)(-4.08546818) + (.00000003)(16.69105025) - (.00045519)(.93475525)$$

$$e = 0.10011028$$

$$a = q/1-e = 1.053706/.89988972$$

Comparing the above results it is apparent that the position of the orbital plane is correct but that there is a discrepancy in the orientation of the orbit in its plane and in the size and shape of the elliptical orbit. Reviewing the computations, it can be seen that the auxiliary parameter p, which has a value of approximately 0.830, is too small. In the case of an elliptical orbit it should be slightly greater than one. Since p is a function of the auxiliary parameters u_j, it is believed that the source of the discrepancy rests here. This would affect the results in the computations for σ_j and for the argument of the perigee, w, also. To avoid continuing the problem with this discrepancy present I decided to use the published orbital data from the 82 observations rather than to continue with the non-realistic values for e and a derived from the computations. The orbital parameters computed for the times of observations in the next chapter are based on those published in Smithsonian Astrophysical Observatory Special Report #78.

5. DETERMINATION OF THE GEOGRAPHIC COORDINATES OF THE UNKNOWN STATION.

There are several methods available to determine the position of an unknown station from observations of a celestial body. Two of these methods were devised by William Markowitz of the U. S. Naval Observatory for his well known Moon Camera Method and can be adopted for use with artificial earth satellites. These methods are called the linear, which gives the coordinates of an unknown station, and the differential, which provides a way to find the corrections to an assumed approximate position. The differential method has been selected for use in this particular problem. As mentioned earlier in this paper, only two approximations in the differential method will be made.

The procedure to determine the coordinates of the unknown station is as follows:

1. The orbital elements of the satellite are known for a particular time, T_0, the time of epoch. Now assuming these elements are correct, the orbital elements are then determined for the times of the observations from the unknown station.

2. Using these elements the geocentric right ascension, geocentric declination, and the range of the satellite from the earth's center, will be predicted for the times of the observations from the unknown station.

3. The differences in the computed coordinates and the observed coordinates of the satellite will furnish us with a set of coefficients for three differential equations to be solved for each observation. The solution of the differential equations will produce the increments $\Delta\alpha'$ and $\Delta\delta'$ which are added to the approximate coordinates. Using these

new coordinates, the differential process will be repeated and the resultant increments will be added to the approximate coordinates again. This will be the final result.

Refer to Figure 6.

The observer, O, views the satellite along the line OS' at a distance of r. The coordinates observed from this position are the apparent right ascension, α', and the apparent declination, δ'. From the center of the earth, E, the satellite would be seen along the line ES' at a range of R. The coordinates observed from this position are the geocentric right ascension, α, and the geocentric declination, δ. The coordinates of the satellite derived from the orbital elements are the geocentric coordinates and will differ from the observed coordinates determined from the camera film by the amount of arc, SS', in the orbital plane. This difference enables us to determine the observer's position on the surface of the earth. This difference between the computed and the observed positions of the satellite should be very small but not so small that the two positions would be at the zenith of the observer. At this position both the computed and observed positions would have the same value and the observer's position could not be determined. Another indeterminate situation would exist if the satellite was observed at the horizon. Here the parallactic angle would be too large and the effect of refraction would make the observations useless. [12, p.34].

Proceeding with the problem, the following data are known:

Location of Observer:	Curacao, N.W.I.
Approximate Position:	$\phi = 12^\circ\, 13'23''.21$ N
	$\lambda = 68^\circ\, 55'07''.86$ W
	$x_a = 0.351992842$
	$y_a = 0.913105082$
	$z_a = 0.210569117$
Satellite Observed:	Explorer I (1958 Alpha)

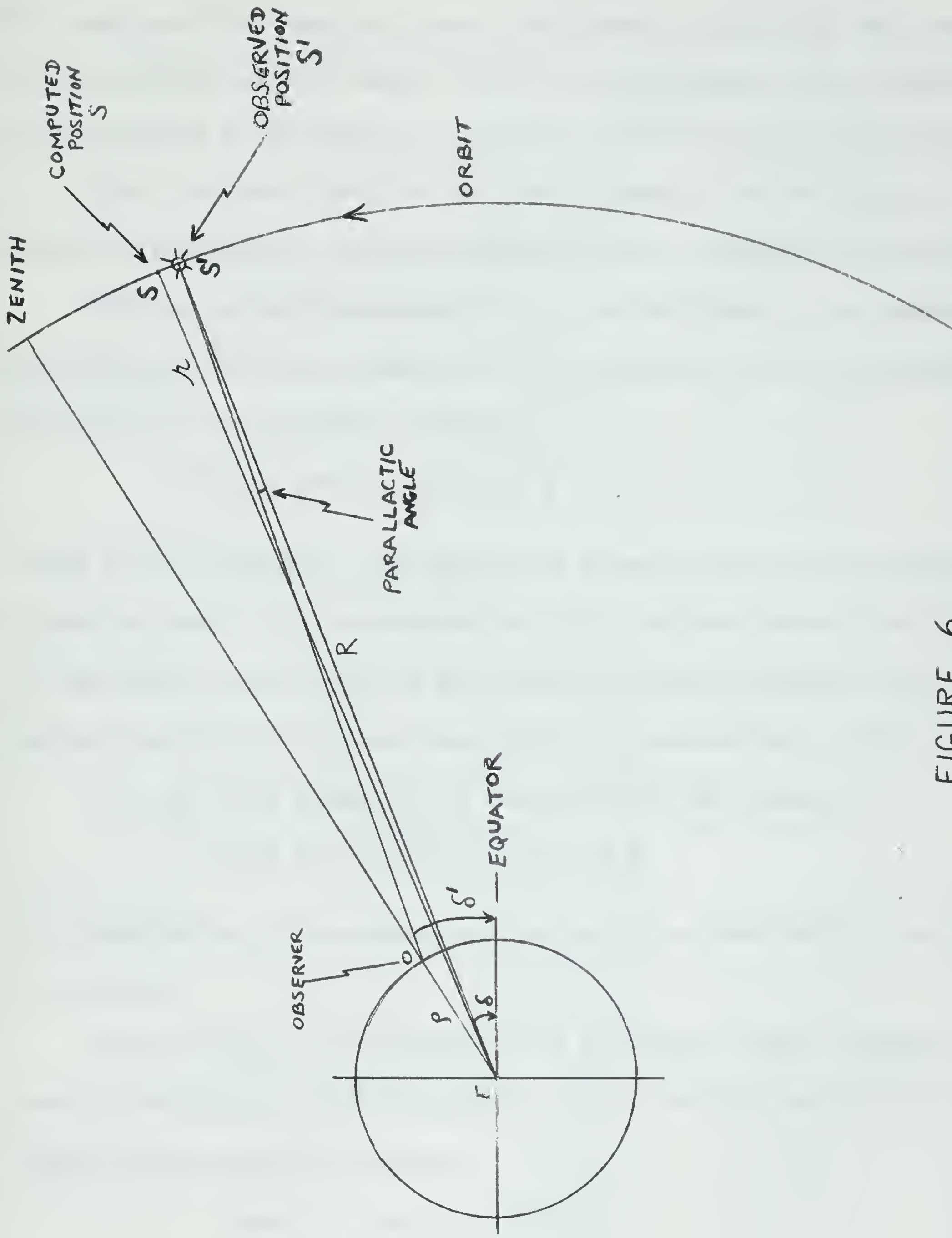

ZENITH
COMPUTED POSITION S
OBSERVED POSITION S'
ORBIT
S
S'
r
PARALLACTIC ANGLE
R
EQUATOR
δ'
OBSERVER
O
ρ
δ
E

FIGURE 6

Universal Time of Observations: $23^h51^m47^s.04$
$23^h51^m54^s.98$
$23^h52^m12^s.45$

The approximate rectangular space coordinates, x_a, y_a, z_a, are found in the same manner as that used to find the coordinates of the known station. This procedure is discussed in Chapter 4 and will not be shown here.

Using the above data, the orbital parameters for the times of observations are computed. These computations are contained in Table 12.

With the orbital parameters known, the next step is to compute the eccentric and true anomalies of the satellite using this formula [4,p.161] for the eccentric anomaly:

$$E = M + e \sin M + \frac{e^2}{2} \sin 2M + \dots \dots$$

where M is in radians. The additional terms in this formula are not listed or used in the determination of E. Moulton states that because of the small eccentricity of the orbit the series converges very rapidly and the first three terms give an accurate value of E.

Then the true anomaly V is computed from the formula:

$$\tan \tfrac{1}{2} V = \sqrt{(1+e)/(1-e)} \tan \frac{E}{2}$$

The computations of the anomalies for the three observations are listed in Table 13.

The next step is to determine the geocentric right ascension, geocentric declination, and geocentric distance of the satellite for each observation using the formulae:

$$\sin \delta = \sin i \sin (w + V)$$

$$\alpha = \Omega + \arctan \left[\cos i \tan (w + V)\right]$$

$$R = a(1 - e \cos E)$$

where a is the semi-major axis of the elliptical orbit in terms of the

TABLE 12

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
Location	Curacao,N.W.I.	Curacao,N.W.I.	Curacao,N.W.I.
Time of Observation (U.T.)	$23^h51^m47^s.04$	$23^h51^m54^s.98$	$23^h52^m12^s.450$
Date of Observation	18 December 1960	18 December 1960	18 December 1960
Smithsonian Date of Observation(t)	37286.994294444	37286.994386343	37286.994588542
Smithsonian Date of Epoch (T_o)	37291.0	37291.0	37291.0
$(t-T_o)$	-4.005705556	-4.005613657	-4.005411458
w (Mean)	111°23'27".7255	111°23'30".1299	111°23'35".4200
sin w	.93111290	.93110865	.93109929
cos w	-.36473110	-.36474195	-.36476583
$(t-T_o)^2$	16.04567700	16.04494077	16.04332095
$(t-T_o)^3$	-64.27425751	-64.26983387	-64.26010156

$$w = (140^\circ.6 \pm 2) + 7^\circ.275(t-T_o) + 0^\circ.000427(t-T_o)^2 + 0^\circ.224 \cos w$$

$7^\circ.275(t-T_o)$	-29°.14150792	-29°.14083935	-29°.13936836
$0^\circ.000427(t-T_o)^2$	0°.00685150	0°.00685119	0°.00685050
$0^\circ.224 \cos w$	-0°.08169977	-0°.08170220	-0°.08170755
w_{t_o}	140°.6 ± 2	140°.6 ± 2	140°.6 ± 2
w_t	111°.38364381	111°.38430964	111°.38577459
w_t	111° 23'01".118	111°23'03".515	111°23'08".789

$$\Omega = (186^\circ.278 \pm 9) - (4^\circ.896(t-T_o) - 0^\circ.000298(t-T_o)^2 + 0^\circ.00887 \cos w$$

$4^\circ.896(t-T_o)$	19°.61193440	19°.61148446	19°.61049450
$0^\circ.000298(t-T_o)^2$	-0°.00478161	-0°.00478139	-0°.00478091
$0^\circ.00887 \cos w$	-0°.00323516	-0°.00323526	-0°.00323547
Ω_{T_o}	186°.278 ± 9	186°.278 ± 9	186°.278 ± 9
Ω_t	205°.88191763	205°.88146781	205°.88047812
Ω_t	205°52'54".903	205°52'53".284	205°52'49".721

$$i = (33^\circ.210 \pm 3) - 0^\circ.000554(t-T_o) + 0^\circ.50\text{x}10^{-5}(t-T_o)^2 - 0^\circ.00473 \sin w$$

$0^\circ.000554(t-T_o)$	0°.00221916	0°.00221911	0°.00221900
$0^\circ.000005(t-T_o)^2$	0°.00008023	0°.00008022	0°.00008022
$0^\circ.00473 \sin w$	-0°.00440416	-0°.00440414	-0°.00440410
i To	33°.210 ± 3	33°.210 ± 3	33°.210 ± 3
i_t	33°.20789523	33°.20789519	33°.20789512
it	33°12'28".423	33°12'28".423	33°12'28".422

$$e = (.09971 \pm 8) - 0.000202(t-T_o) + 0.30\text{x}10^{-7}(t-T_o)^2 + 0.00045519 \sin w$$

$0.000202(t-T_o)$	0.00080915	0.00080913	0.00080909
$0.00000003(t-T_o)^2$	0.00000048	0.00000048	0.00000048
$0.00045519 \sin w$	0.00042383	0.00042383	0.00042383
e_{T_o}	0.09971 ± 8	0.09971 ± 8	0.09971 ± 8
e_t	0.10094346	0.10094344	0.10094340

(continued on page 49)

$$M = (.94016) + (13.435134)(t-T_o) + 0.0002839(t-T_o)^2 - 0.11\times10^{-5}(t-T_o)^3 - 0.000631\times10^{-3}\cos w$$

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
$(13.435134)(t-T_o)$	-53.81719091	-53.81595623	-53.81323966
$(0.0002839)(t-T_o)^2$	0.00455537	0.00455516	0.00455470
$(0.0000011)(t-T_o)^3$	0.00007070	0.00007070	0.00007069
$(0.00000063)\cos w$	0.00000023	0.00000023	0.00000023
M_{T_o}	0.94016	0.94016	0.94016
M	-53.81256461	-53.81133014	-53.80861404
M_t (Revolutions)	0.12759539	0.12882986	0.13154596
M_t	45°.93434040	46°.37874960	47°.35654560
M_t	45°56'03".625	46°22'43".499	47°21'23".564

TABLE 13

COMPUTATION OF ECCENTRIC AND TRUE ANOMALIES

$$E = M + e \sin M + \frac{e^2}{2} \sin 2M$$

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
e^2	0.01018958	0.01018958	0.01018957
$e^2/2$	0.00509479	0.00509479	0.00509479
$1 + e$	1.10094346	1.10094344	1.10094340
$1 - e$	0.89905654	0.89905656	0.89905660
$(1+e)/(1-e)$	1.22455419	1.22455414	1.22455404
$\left[(1+e)/(1-e)\right]^{\frac{1}{2}}$	1.10659577	1.10659574	1.10659570
sin M	0.71854327	0.72391604	0.73558351
sin 2M	0.99946819	0.99884211	0.99661864
e sin M	0.07253224	0.07307458	0.07425230
$(e^2/2)\sin 2M$	0.00509208	0.00508889	0.00507756
E (radians)	0.87932980	0.88762535	0.90585750
E	50°.38188634	50°.85718634	51°.90181159
E	50°22'54".791	50°51'25".871	51°54'06".522
E/2	25°11'27".396	25°25'42".986	25°57'03".261
tan E/2	0.47037122	0.47544692	0.48667234
$\left[(1+e)/(1-e)\right]^{\frac{1}{2}}\tan E/2 = \tan V/2$	0.52051080	0.52612754	0.53854952
V/2	27°29'50".872	27°45'00".338	28°18'16".789
V	54°59'41".744	55°30'00".676	56°36'33".578

[illegible]

TABLE 14

COMPUTATION OF GEOCENTRIC DECLINATION, GEOCENTRIC RIGHT ASCENSION, AND GEOCENTRIC DISTANCE OF THE SATELLITE.

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
$\sin \delta = \sin i \sin (w+V)$			
$(w+V)$	166°22'42".862	166°53'04".191	167°59'42".367
$\sin (w+V)$	0.23550558	0.22691483	0.20799531
$\sin i$	0.54767853	0.54767853	0.54767853
$\sin \delta$	0.12898135	0.12427638	0.11391457
δ	7°24'38".636	7°08'20".290	6°32'27".682
$\alpha = \Omega + \arctan \left[\cos i \tan (w+V)\right]$			
$\tan (w+V)$	-0.24232136	-0.23299254	-0.21264591
$\cos i$	0.83668886	0.83668886	0.83668886
$\cos i \tan (w+V)$	-0.20274758	-0.19494226	-0.17791846
Ω	205°52'54".903	205°52'53".284	205°52'49".721
$\arctan \left[\cos i \tan(w+V)\right]$	168°32'19".601	168°58'08".322	169°54'41".720
α	14°25'14".504	14°51'01".606	15°47'31".441
$a = q/(1-e)$			
q (M_m)	6.72127351	6.72127349	6.72127349
q (earth's radius)	1.05375739	1.05375739	1.05375738
a	1.17207021	1.17207018	1.17207012
$R = a(1 - e \cos E)$			
e	0.10094346	0.10094344	0.10094340
E	50°22'54".791	50°51'25".871	51°54'06".522
cos E	0.63766754	0.63125552	0.61701099
e cos E	0.06436837	0.06372110	0.06228319
1 - e cos E	0.93563163	0.93627890	0.93771681
R	1.09662596	1.09738458	1.09906985

earth's radius. The value "a" can be computed from the formula:

$$a = q/(1-e)$$

where q is the distance of the satellite at perigee. The computation of these values are shown in Table 14. θ_g, the Greenwich hour angle of the vernal equinox, is tabulated in Table 15. The computations of the sidereal times of observations from which this value is derived is not shown or listed. The sidereal time was established in the same manner as it was for the observations from the known station.

Now the computed apparent right ascension and declination must be found using the formulae [12, p.37] :

$$\tan(\alpha'_c - \alpha) = \frac{x_a \sin G + y_a \cos G}{-x_a \cos G + y_a \sin G + R \cos \delta}$$

and

$$\tan \delta'_c = \frac{\cos(\alpha'_c - \alpha)(R \sin \delta - z_a)}{-x_a \cos G + y_a \sin G + R \cos \delta}$$

Then the differences between the computed and the observed values are determined by:

$$\Delta \alpha' = \alpha' - \alpha'_c$$

$$\Delta \delta' = \delta' - \delta'_c$$

The values and computations of δ'_c and α'_c and the increments $\Delta \alpha'$ and $\Delta \delta'$ are tabulated in Tables 15 and 16.

The differential equations to be solved [12,p.36] are:

$$a_1\, dx_a + a_2\, dy_a = a_3\, d\alpha'$$

$$b_1\, dx_a + b_2\, dy_a + b_3\, dz_a = b_4\, d\alpha' + b_5\, d\delta'$$

The coefficients of these equations are:

$$a_1 = \sin G + \tan(\alpha' - \alpha) \cos G$$

$$a_2 = \cos G - \tan(\alpha' - \alpha) \sin G$$

$$a_3 = \sec^2(\alpha' - \alpha)(R \cos \delta - x_a \cos G + y_a \sin G)$$

$$b_1 = \tan \delta' \cos G$$
$$b_2 = - \tan \delta' \sin G$$
$$b_3 = - \cos (\alpha' - \alpha)$$
$$b_4 = \sin (\alpha' - \alpha) \quad (R \sin \delta - z_a)$$
$$b_5 = \sec^2 \delta' (R \cos \delta - x_a \cos G + y_a \sin G)$$

The computation of these coefficients are contained in Table 16. The solution of the first approximation differential equations is shown in Table 17. ACIC Technical Report No. 86 suggests these equations be solved by Doolittle's method but being more familiar with the Gaussian solution of normal equations, I decided to use the latter method. The solution of the normal and reduced equations yields the increments Δx_a, Δy_a, and Δz_a. These increments are then added to the original approximate coordinates to find the corrected position of the observer. The corrected space coordinates are:

x_o	y_o	z_o
+ 0.351462753	+ 0.912606236	+ 0.210920824

Since the International Ellipsoid was used as the reference ellipsoid, the x_a, y_a, z_a coordinates can be multiplied by 6,378,388 meters, the semi-major axis of the ellipsoid, to find the metric distance of ρ, the radius of the observer's position from the earth's center. This has not been done because it is of little interest in this problem. However, the distance of the observer from the earth's center is found in units of the earth's radius by the equation:

$$\rho = \sqrt{x_o^2 + y_o^2 + z_o^2} = 1.000431808$$

Then the resultant geographic coordinates of the observer may be easily derived as follows:

$$\sin \phi' = z^o/\rho = + 0.21082979$$
$$\tan \phi' = + 0.21567762$$

TABLE 15

SOLUTION FOR COMPUTED (APPARENT RIGHT ASCENSION, α'_c) (APPARENT DECLINATION, δ'_c)

FIRST APPROXIMATION

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
α	14°25'14".504	14°51'01".606	15°47'31".441
δ	7°24'38".636	7°08'20".290	6°32'27".682
R	1.09662596	1.09738458	1.09906985
α'	355° 46'30".	359° 39' 00".	8° 03' 00".
δ'	-30° 51'00".	-32° 52' 00".	-36° 39' 00".
θ_g (GHA γ)	85°31'54".975	85°33'54".405	85°38'17".160
$\alpha - \theta_g = G$	-71°06'40".471	-70°42'52".799	-69°50'45".719
sin G	-0.94614889	-0.94388553	-0.93877014
cos G	0.32373178	0.33027279	0.34454407
sin δ	0.12898135	0.12427638	0.11391457
cos δ	0.99164703	0.99224765	0.99349055
$(\alpha' - \alpha)$	341°21'15".496	344°47'58".394	-7°44'31".441
sin $(\alpha' - \alpha)$	-0.31971509	-0.26219670	-0.13471374
cos $(\alpha' - \alpha)$	0.94751372	0.96501495	0.99088456
tan $(\alpha' - \alpha)$	-0.33742529	-0.27170236	-0.13595301
sec$^2(\alpha' - \alpha)$	1.11385584	1.07382217	1.01848322
tan δ'	-0.59730303	-0.64610406	-0.74402040
sec^2 δ'	1.35677091	1.41745046	1.55356636

$$\tan(\alpha'_c - \alpha) = \frac{x_a \sin G + y_a \cos G}{-x_a \cos G + y_a \sin G + R\cos\delta}$$

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
x_a	$\longrightarrow$	+ 0.3519992842	$\longleftarrow$
y_a	$\longrightarrow$	+ 0.913105082	$\longleftarrow$
z_a	$\longrightarrow$	+ 0.210569117	$\longleftarrow$
x_a sin G	-0.33303764	-0.33224095	-0.33044037
y_a cos G	0.29560113	0.30157376	0.31460494
$-x_a$ cos G	-0.11395127	-0.11625366	-0.12127705
y_a sin G	-0.86393336	-0.86186667	-0.85719579
R cos δ	1.08746588	1.08887727	1.09191551
$x_a \sin G + y_a \cos G$	-0.03743651	-0.03066719	-0.01583543
$-x_a \cos G + y_a \sin G + R\cos\delta$	0.10958125	0.11075694	0.11344267
tan $(\alpha'_c - \alpha)$	-0.341632244	-0.27688730	-0.13958972
$(\alpha'_c - \alpha)$	341°08'17".406	344°31'23".757	-7°56'47".593
α'_c	355°33'31".910	359°22'25".363	7°50'43".848

$$\tan \delta'_c = \frac{\cos(\alpha'_c - \alpha)(R \sin\delta - z_a)}{-x_a \cos G + y_a \sin G + R\cos}$$

[illegible]

[illegible]

TABLE 15 (Continued)

FIRST APPROXIMATION

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
$\cos(\alpha'_c - \alpha)$	0.94630093	0.96373889	0.99039746
$R \sin \delta$	0.14144430	0.13637898	0.12520007
$(R \sin \delta - z_a)$	-0.06912482	-0.07419014	-0.08536905
$-x_a \cos G + y_a \sin G + R \cos \delta$	0.10958125	0.11075694	0.11344267
$\cos(\alpha'_c - \alpha) \times (R \sin \delta - z_a)$	-0.06541288	-0.07149992	-0.08454929
$\tan \delta'_c$	-0.59693497	-0.64555702	-0.74530413
δ'_c	-30°50'04".037	-32°50'40".376	-36°41'50".335

TABLE 16

COMPUTATION OF COEFFICIENTS OF DIFFERENTIAL EQUATIONS

FIRST APPROXIMATION

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
	$a_1 = \sin G + \tan(\alpha' - \alpha)\cos G$		
$\alpha' - \alpha_c = \Delta\alpha'$	0°13'58".090	0°16'34".637	0°12'16".152
$\Delta\alpha'$	0°.23280278	0°.27628805	0°.20448666
$\Delta\alpha'$ (radians)	0.00406318	0.00482214	0.00356897
$\delta' - \delta_c' = \Delta\delta'$	-0°01'04".037	-0°01'19".624	0°02'50".335
$\Delta\delta'$	-0°.01778806	-0°.02211778	0°.04731527
$\Delta\delta'$ (radians)	-0.00031046	-0.00038603	0.00082581
sin G	-0.94614889	-0.94388553	-0.93877014
$\tan(\alpha' - \alpha)\cos G$	-0.10923529	-0.08973590	-0.04684180
a_1	-1.05538418	-1.03362143	-0.98561194
	$a_2 = \cos G - \tan(\alpha' - \alpha)\sin G$		
cos G	0.32373178	0.33027279	0.34454407
$\tan(\alpha' - \alpha)\sin G$	0.31925456	0.25645593	0.12762863
a_2	0.00447722	0.07381686	0.21691544
	$a_3 = \sec^2(\alpha' - \alpha)(R\cos\delta - x_a\cos G + y_a\sin G$		
$\sec^2(\alpha' - \alpha)$	1.11385584	1.07382217	1.01848322
$R\cos\delta$	1.08746588	1.08887727	1.09191551
$x_a\cos G$	0.11395127	0.11625366	0.12127705
$y_a\sin G$	-0.86393336	-0.86186667	-0.85717579
$R\cos\delta - x_a\cos G + y_a\sin G$	0.10958125	0.11075694	0.11344267
a_3	0.12205772	0.11893326	0.11553946
	$b_1 = \tan\delta'\cos G$		
b_1	-0.19336597	-0.21339059	-0.25634782
	$b_2 = -\tan\delta'\sin G$		
b_2	-0.56513760	-0.60984827	-0.69846414
	$b_3 = -\cos(\alpha' - \alpha)$		
b_3	-0.94751372	-0.96501445	-0.99088456
	$b_4 = \sin(\alpha' - \alpha)(R\sin\delta - z_a)$		
$\sin(\alpha' - \alpha)$	-0.31971509	-0.26219670	-0.13471374
$R\sin\delta$	0.14144430	0.13637898	0.12520007
z_a	0.21056912	0.21056912	0.21056912
$R\sin\delta - z_a$	-0.06912482	-0.07419014	-0.08536905
b_4	0.02210025	0.01945241	0.01150038

[illegible]

TABLE 16 (Continued)

FIRST APPROXIMATION

$$b_5 = \sec^2 \delta' (R \cos \delta - x_a \cos G + y_a \sin G)$$

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
$\sec^2 \delta'$	1.35677091	1.41745046	1.55356636
$R \cos \delta - x_a \cos G + y_a \sin G$	0.10958125	0.11075694	0.11344267
b_5	0.14867665	0.15699248	0.17624072

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

TABLE 17

SOLUTION OF DIFFERENTIAL EQUATIONS

FIRST APPROXIMATION

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
$a_3 \Delta\alpha'$	0.00049594	0.00057351	0.00041236
$b_4 \Delta\alpha'$	0.00008980	0.00009380	0.00004104
$b_5 \Delta\delta'$	-0.00004616	-0.00006060	0.00014554
$b_4 \Delta\alpha' + b_5 \Delta\delta'$	0.00004364	0.00003320	0.00018658

OBSERVATION EQUATIONS

$$a_1 dx_a + a_2 dy_a = a_3\, d\alpha'$$
$$b_1 dx_a + b_2 dy_a + b_3\, dz_a = b_4\, d\alpha' + b_5 d\delta'$$

dx_a	dy_a	dz_a	ℓ	S
-1.05538418	0.00447722		-0.00049594	-1.05140290
-0.19336597	-0.56513760	-0.94751372	-0.00004364	-1.70606093
-1.03362143	0.07381686		-0.00057351	-0.96037808
-0.21339059	-0.60984827	-0.96501445	-0.00003320	-1.78828651
-0.98561194	0.21691544		-0.00041236	-0.76910886
-0.25634782	-0.69846414	-0.99088456	-0.00018658	-1.94588310

NORMAL EQUATIONS

dx_a	dy_a	dz_a	ℓ	Σ
3.302280071	0.123645699	0.643153009	0.001585979	4.070664758
	1.231668857	1.816085355	0.000041227	3.171441137
		2.810887350	0.000258267	5.270383981
			0.000000783	0.001886255

REDUCED EQUATIONS

	dx_a	dy_a	dz_a	ℓ	Σ
dx_a	3.302280071	0.123645699	0.643153009	0.001585979	4.070664758
dy_a	-0.037442523	1.227039250	1.792004084	-0.000018156	3.019025178
dz_a	-0.194760285	-1.460429309	0.068531401	-0.000024103	0.068507298
	-0.000480268	0.000014797	0.000351707	0.000000013	0.000000012

dx_a	dy_a	dz_a
-0.000530089	-0.000498846	+0.000351707

x_a	0.351992842	y_a	0.913105082	z_a	0.210569117
Δx_a	-0.000530089	Δy_a	-0.000498846	Δz_a	0.000351707
x_o	+ 0.351462753	y_o	+ 0.912606236	z_o	+ 0.210920824

[illegible]

$$\tan \phi = \tan \phi'/1-e^2 = 0.21567762/0.99327733 = 0.21713736$$

$$\tan \lambda = y_o/x_o = 2.59659446$$

which result in:

$$\phi = 12^\circ\ 15'\ 03''.168\ N$$

$$\lambda = 68^\circ\ 56'\ 14''.336\ W$$

The differential method was repeated again for a second approximation. The corrected space coordinates from the first differential process were used as the approximate observer's position. The data used in and the computations of the second differential approximation are listed in Tables 18, 19, and 20.

The corrected and final results of the second approximation are:

x_o	y_o	z_o
+ 0.351483504	+ 0.912709058	+ 0.210849193

$$\rho = \sqrt{x_o^2 + y_o^2 + z_o^2} = 1.000517795$$

$$\sin \phi' = z_o/\rho = 0.21074007$$

$$\tan \phi' = 0.21558157$$

$$\tan \phi = \tan \phi'/1-e^2 = 0.21558157/0.99327733 = 0.21704066$$

$$\tan \lambda = y_o/x_o = 2.59673370$$

The corrected and final geographic coordinates are:

$$\phi = 12^\circ\ 14'\ 44''.118\ N$$

$$\lambda = 68^\circ\ 56'\ 18''.045\ W$$

Further approximations would probably yield changes to these coordinates but as stated previously the assumption that two approximations are sufficient has been made.

As a note of interest, and to satisfy my own curiosity, further calculations were made using the linear method but are not included in this paper. First, I used my own computed value of the Gaussian gravitational constant, k, in the formula to determine "a":

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

TABLE 18

SOLUTION OF COMPUTED {(APPARENT RIGHT ASCENSION, α'_c / (APPARENT DECLINATION, δ'_c}

SECOND APPROXIMATION

$$\tan(\alpha'_c - \alpha) = \frac{x_a \sin G + y_a \cos G}{-x_a \cos G + y_a \sin G + R\cos\delta}$$

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
x_a	→	+ 0.351462753	←
y_a	→	+ 0.912606236	←
z_a	→	+ 0.210920824	←
$x_a \sin G$	-0.33253609	-0.33174060	-0.32994274
$y_a \cos G$	0.29543964	0.30140901	0.31443307
$-x_a \cos G$	-0.11377966	-0.11607858	-0.12109441
$y_a \sin G$	-0.86346138	-0.86139582	-0.85672749
$R \cos\delta$	1.08746588	1.08887727	1.09191551
$x_a \sin G + y_a \cos G$	-0.03709645	-0.03033159	-0.01550967
$-x_a \cos G + y_a \sin G + R\cos\delta$	0.11022484	0.11140287	0.11409361
$\tan(\alpha'_c - \alpha)$	-0.33655254	-0.27226938	-0.13593811
$(\alpha'_c - \alpha)$	341°23'57".123	344°46'09".492	-7°44'28".424
α'_c	355°49'11".627	359°37'11".098	8° 03'03".017

$$\tan\delta'_c = \frac{\cos(\alpha'_c - \alpha)(R\sin\delta - z_a)}{-x_a \cos G + y_a \sin G + R\cos}$$

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
$\cos(\alpha'_c - \alpha)$	0.94776396	0.96487589	0.99088653
$R \sin\delta$	0.14144430	0.13637898	0.12520007
$R \sin\delta - z_a$	-0.06947652	-0.07454184	-0.08572075
$\cos(\alpha'_c - \alpha)(R\sin\delta - z_a)$	-0.06584734	-0.07192362	-0.08493954
$\tan\delta'_c$	-0.59739111	-0.64561730	-0.74447237
δ'_c	-30°51'13".391	-32°50'49".152	-36°39'59".995

[illegible]

TABLE 19

COMPUTATION OF COEFFICIENTS OF DIFFERENTIAL EQUATIONS

SECOND APPROXIMATION

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
$a_1 = \sin G + \tan(\alpha' - \alpha)\cos G$			
$\alpha' - \alpha'_c = \Delta\alpha'$	$-0^\circ 02' 41''.627$	$0^\circ 01' 48''.912$	$-0^\circ 00' 03''.017$
$\Delta\alpha'$	$-0^\circ.04484639$	$0^\circ.03025333$	$-0^\circ.00083806$
$\Delta\alpha'$ (radians)	-0.00078272	0.00052802	-0.00001463
$\delta' - \delta'_c = \Delta\delta'$	$0^\circ 00' 13''.391$	$-0^\circ 01' 10''.848$	$0^\circ 00' 59''.995$
$\Delta\delta'$	$0^\circ.00371972$	$-0^\circ.01968000$	$0^\circ.01666528$
$\Delta\delta'$ (radians)	0.00006492	-0.00034381	0.00029086
a1	-1.05538418	-1.03362143	-0.98561194
$a_2 = \cos G - \tan(\alpha' - \alpha)\sin G$			
a_2	0.04447722	0.07381686	0.21691544
$a_3 = \sec^2(\alpha' - \alpha)(R\cos\delta - x_a\cos G + y_a\sin G)$			
$\sec^2(\alpha' - \alpha)$	1.11385584	1.07382217	1.01848322
$R\cos\delta$	1.08746588	1.08887727	1.09191551
$x_a\cos G$	0.11377966	0.11607858	0.12109441
$y_a\sin G$	-0.86346138	-0.86139582	-0.85672749
$(R\cos\delta - x_a\cos G + y_a\sin G)$	0.11022484	0.11140287	0.11409361
a_3	0.12277458	0.11962687	0.11620243
$b_1 = \tan\delta'\cos G$			
b_1	-0.19336597	-0.21339059	-0.25634782
$b_2 = -\tan\delta\sin G$			
b_2	-0.56513760	-0.60984827	-0.69846414
$b_3 = -\cos(\alpha' - \alpha)$			
b_3	-0.94751372	-0.96501445	-0.99088456
$b_4 = \sin(\alpha' - \alpha)(R\sin\delta - z_a)$			
$\sin(\alpha' - \alpha)$	-0.31971509	-0.26219670	-0.13471374
$R\sin\delta$	0.14144430	0.13637898	0.12520007
z_a	0.21092082	0.21092082	0.21092082
$R\sin\delta - z_a$	-0.06947652	-0.07454184	-0.08572075
b_4	0.02221269	0.01954462	0.01154776

[illegible]

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
	$b_5 = \sec^2 \delta'(R \cos \delta - x_a \cos G + y_a \sin G)$		
$\sec^2 \delta'$	1.35677091	1.41745046	1.55356636
$(R \cos \delta - x_a \cos G + y_a \sin G)$	0.11022484	0.11140287	0.11409361
b_5	0.14954986	0.15790805	0.17725199

SOLUTION OF DIFFERENTIAL EQUATIONS

SECOND APPROXIMATION

	OBSERVATION 1	OBSERVATION 2	OBSERVATION 3
$a_3 \Delta\alpha'$	-0.00009610	0.00006317	-0.00000170
$b_4 \Delta\alpha'$	-0.00001739	0.00001032	-0.00000170
$b_5 \Delta\delta'$	0.00000971	-0.00005429	0.00005155
$b_4 \Delta\alpha' + b_5 \Delta\delta'$	-0.00000768	-0.00004397	0.00005138

OBSERVATION EQUATIONS

$$a_1\, dx_a + a_2\, dy_a = a_3\, d\alpha'$$
$$b_1\, dx_a + b_2\, dy_a + b_3\, dz_a = b_4\, d\alpha' + b_5\, d\delta'$$

dx_a	dy_a	dz_a	ℓ	S
-1.05538418	0.00447722		0.00009610	-1.05081086
-0.19336597	-0.56513760	-0.94751372	0.00000768	-1.70600961
-1.03362143	0.07381686		-0.00006317	-0.95986774
-0.21339059	-0.60984827	-0.96501445	0.00004397	-1.78820934
-0.98561194	0.21691544		0.00000170	-0.76869480
-0.25634782	-0.69846414	-0.99088456	-0.00005138	-1.94574790

NORMAL EQUATIONS

dx_a	dy_a	dz_a	ℓ	Σ
3.302280071	0.123645699	0.643153009	-0.000035501	4.069043279
	1.231668857	1.816085355	0.000000868	3.171400778
		2.810887350	0.000001203	5.270126916
			0.000000018	-0.000033412

REDUCED EQUATIONS

	dx_a	dy_a	dz_a	ℓ	Σ
dx_a	3.302280071	0.123645699	0.643153009	-0.000035501	4.069043279
dy_a	-0.037442523	1.227039250	1.792004084	0.000002197	3.019045531
dz_a	-0.194760285	-1.460429309	0.068531401	0.000004909	0.068536309
	0.000010750	-0.000001790	-0.000071631	0.000000017	0.000000016

dx_a	dy_a	dz_a
+0.000020851	+0.000102822	-0.000071631

x_a	0.351462753	y_a	0.912606236	z_a	0.210920824
Δx_a	+0.000020851	Δy_a	+0.000102822	Δz_a	-0.000071631
x_o	+0.351483504	y_o	+0.912709058	z_o	+0.210849193

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\sqrt{a^3} = \frac{k}{n}$$

where n is the velocity of the mean anomaly, M. The resultant "a" from this formula has the value 1.17190297 in contrast with the value 1.17207018 which was computed from the formula $a = q/1-e$. The final result of geographic coordinates using this value and the linear method is listed in the summary.

Second, the linear method was used again in which the eccentricity derived from the mean orbital elements was used instead of the smooth value for e. The mean eccentricity for the time of observation is 0.10067497. The resultant coordinates from this experiment are also listed in the summary.

Summary:

Method	Latitude, ϕ	Longitude, λ
Approximate	12° 13' 23".21 N	68° 55' 07".86
Differential (1st)	12° 15' 03".168	68° 56' 14".336
Differential (2nd)	12° 14' 44".118	68° 56' 18".045
Linear (using computed k)	12° 15' 06".351	68° 56' 11".075
Linear (with mean e)	12° 15' 45".276	68° 57' 25".278
Actual	12° 05' 50".4	68° 50' 14".0

6. THE INTER-CONTINENTAL TIE

The last step of this thesis is to effect the "tie" between the two stations, the geographic coordinates of which are now known. This problem can be considered to be one of geometric geodesy or what is more commonly known as the "determination of long lines" and is considered to be a part of higher geodesy.

H. F. Rainsford states there are two main problems in the field of long lines [6, p.12]:

a) Given the geographical coordinates of two points, to find the distance and azimuths between them; commonly called the reverse (or inverse) problem;

b) Given the geographical coordinates of one point and the distance and azimuth from it to another, to find the coordinates of the second point; commonly called the direct problem.

Since only the geographic coordinates of the two points are known in this case, the reverse problem applies.

The following data are given:

Point 1	Point 2
Curacao, N. W. I.	Olifantsfontein, South Africa
ϕ_1 12° 14' 44".118 N	ϕ_2 25° 57' 34".700 S
λ_1 68° 56' 18".045 W	λ_2 28° 14' 51".100 E

Problem: Find the forward azimuth, α_{12}, at Point 1; find the back azimuth, α_{21}, at Point 2; and determine the distance, s, between the two points.

Since the reference surface chosen for this paper is the International Ellipsoid, the distance, s, between the points will be one of the basic geodetic curves called the geodesic. It is defined as [13, p.2]:

> a line on any mathematically defined geodetic reference surface that defines the shortest distance between any two points on that reference surface. It is geometrically equivalent to a great circle on the sphere. It is a unique line on the reference ellipsoid that contains the normal to the ellipsoid at each point.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Refer to Figure 7. It can be seen that the line, s, connecting the two points, P1 and P_2, contains the normal to the ellipsoid along its entirety. The forward and back azimuths, α_{12} and α_{21} respectively, are shown at the points. The curvature of the geodesic in the diagram is exaggerated for clarity of figure only.

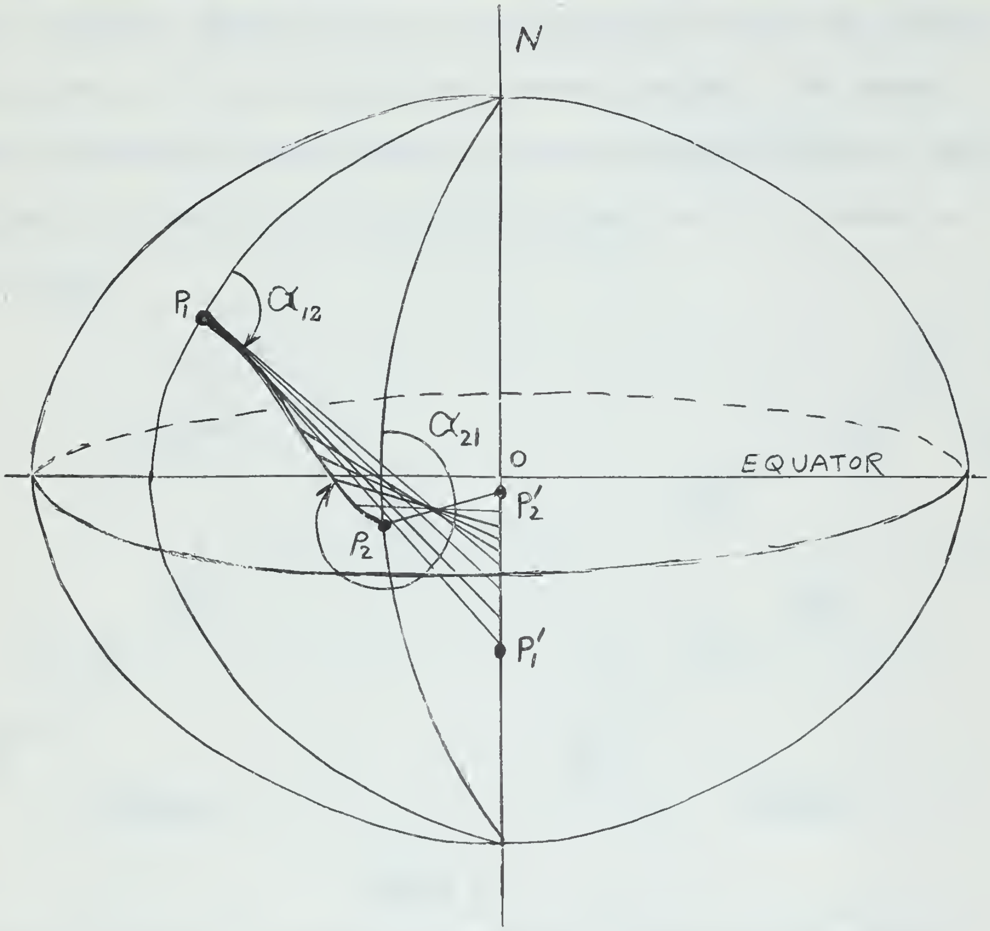
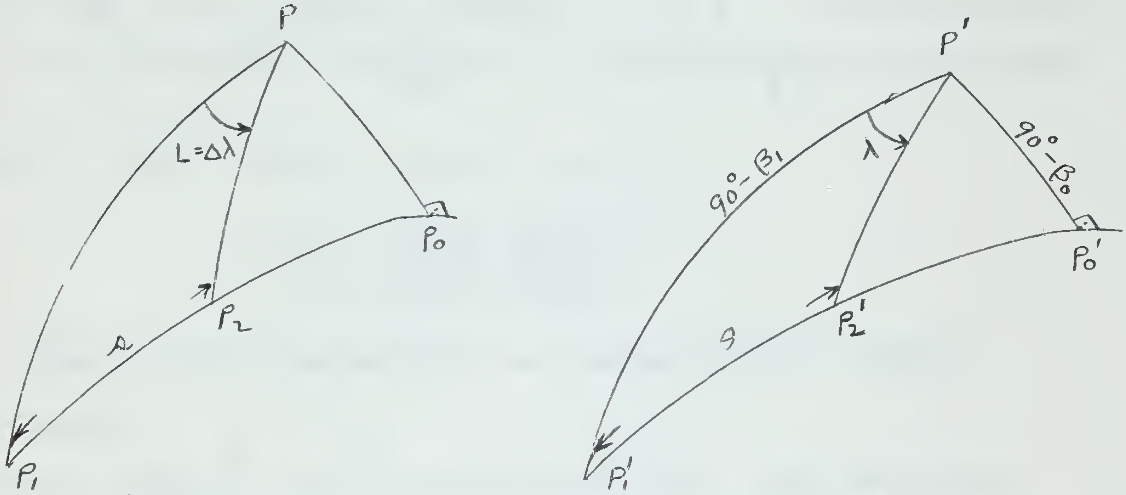


Figure 7.

It should be noted that the distance difference between a geodesic and a plane curve is very small but the difference in the azimuths of the two would be appreciable. As the distance between points increases, the difference between the two lines (geodesic and plane curve) diverges rapidly.

There are several methods available in geometric geodesy to deter-

mine the geodesic and the azimuths but because this particular problem is concerned with the inverse problem and the distance between the two points is approximately 6,000 miles, then a choice of only two methods is available. These are the Rainsford Method and Sodano's Fourth Method. Another method that solves for distances greater than 5,000 miles is Rudoe's Method but since this procedure treats the normal section only, it is not applicable for this problem. The method chosen was Sodano's Fourth which is a non-iterative solution. The accuracy is of the order of $\pm 0''.1$ in azimuth and ± 0.5 meters at 6,000 miles.



Ellipsoid Sphere

Figure 8.

Proceeding with the solution, an auxiliary spherical triangle is employed. In our problem, the westward point, P_1, is Curacao and the eastward point, P_2 is Olifantsfontein, South Africa. The difference in longitude, L, is always positive. [13, p.41].

The first step is to find the parametric latitudes, β_1 and β_2, using the formula

$$\tan \beta = (1 - f) \tan \phi$$

[illegible]

[illegible]

[illegible]

[illegible]

with the results

$$\beta_1 = 12^\circ\ 07'\ 33''.521$$
$$\beta_2 = -25^\circ\ 53'\ 01''.189$$

Using the parametric latitudes, the approximate spherical distance, θ, is found from [13, p.41]:

$$\cos\theta = \sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos L = -0.2017305506$$

and

$$\sin\theta/2 = \left[\cos\beta_1 \cos\beta_2 \sin^2\frac{L}{2} + \sin^2\frac{\beta_2 \quad \beta_1}{2}\right]^{\frac{1}{2}} = 0.7751548582$$

The resulting value for θ is $101^\circ\ 38'\ 17''.336$.

Now λ must be found using the formula $\lambda = L + (\lambda - L)$ where [13, p.42]:

$$(\lambda - L) = \left[\frac{(K_1 + B)\theta + (C+D)\sin\theta + (F+E)G}{K_2}\right] A\rho'' = 0^\circ 18' 20''.866$$

where $\rho = 206264''.80625$. The value for λ is:

$$\begin{aligned} L &= 97^\circ\ 11'\ 09''.145 \\ (L-\lambda) &= \underline{\ 0^\circ\ 18'\ 20''.866} \\ \lambda &= 97^\circ\ 29'\ 30''.011 \end{aligned}$$

Then the forward azimuth α_{12} is computed from the formula [13, p.42]:

$$\cot\alpha_{12} = \left[\tan\beta_2 \cos\beta_1 - \cos\lambda \sin\beta_1\right] / \sin\lambda = 114^\circ 16' 06''.607$$

The back azimuth is [13, p.42]:

$$\cot\alpha_{21} \left[\sin\beta_2 \cos\lambda - \cos\beta_2 \tan\beta_1\right] / \sin\lambda = 277^\circ 49' 56''.504$$

The next step is to find the true spherical distance, θ_o, using $\lambda = 97^\circ 29' 30''.011$ instead of $L = 97^\circ 11' 09''.145$ in the same formula as before:

$$\cos\theta_o = \sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos\lambda = -0.2063866912$$
$$\theta_o = 101^\circ\ 54'\ 38''.472$$

The final step is to find the geodesic distance, s, from the formula [13, p.43]:

[illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$s = b(A_0\, \theta_0 + B_0 \sin \theta_0 \cos 2\sigma - C_0 \sin 2\theta_0 \cos 4\sigma)$$

$s = 11{,}309{,}342.6228$ meters $= 6{,}102.4421$ nautical miles.

Refer to the following tables for the detailed computations of the constants, coefficients, and the unknowns.

Summarizing the results:

Point #1	Point #2
Curacao, N.W.I.	Olifantsfontein, South Africa
ϕ_1 12° 14'44".118 N	ϕ_2 25° 57' 34".700 S
λ_1 68° 56'18".045 W	λ_2 28° 14' 51".100 E

α_{12} = 114° 16' 06".607
α_{21} = 277° 49' 56".504
s = 11,309,342.6228 meters.

This concludes the computations.

TABLE 21

LIST OF CONSTANTS

FOR USE WITH SODANO'S FOURTH METHOD

INTERNATIONAL ELLIPSOID

$1/\alpha = 297.0$
$a = 6,378,388.0000$ meters
$b = 6,356,911.9462$ meters
$e = 0.0819918898$ $\qquad e^2 = 0.0067226700$
$e' = 0.0822688896$ $\qquad e'^2 = 0.0067681702$

$$N = \frac{e'}{e' + e}$$

$$K_1 = \frac{16 N - e'^2}{16e^2N^2 + e'^2} = 237.2388918$$

$$K_2 = \frac{16}{e^2(16e^2N^2 + e'^2} = 70519.51145$$

$$K_3 = \frac{16e^2 N^2 + e'^2}{e'^2} = 4.986520649$$

$$K_4 = \frac{2 e'^2}{16e^2 N^2 + e'^2} = 0.4010812630$$

$$K_5 = \frac{16e^2 N^2}{16e^2N^2 + e'^2} = 0.7994593686$$

$$K_6 = \frac{16 e^2 N^2}{e'^2} = 3.986520649$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

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$$[illegible]$$

$$[illegible]$$

TABLE 22

COMPUTATION OF VARIOUS TERMS

$$A = \frac{\cos\beta_1 \cos\beta_2 \sin L}{\sin\theta}$$

$\cos\beta_1 \cos\beta_2$		0.8796086600		
$\sin L$		0.9921455600		
$\cos\beta_1 \cos\beta_2 \sin L$		0.8726998266		
$(\cos\beta_1 \cos\beta_2 \sin L)/\sin\theta$		0.8910182190	=	A

$$B = A^2 = 0.7939134666$$

$$C = (\cos\theta - B\cos\theta)K_3$$

K_3		4.9865206649		
$B\cos\theta$		-0.1601566007		
$(\cos\theta - B\cos\theta)$		-0.0415739499		
$(\cos\theta - B\cos\theta)K_3$	=	-0.0083372662	=	C

$$D = -K_4 \sin\beta_1 \sin\beta_2$$

K_4		0.4010812630		
$\sin\beta_1 \sin\beta_2$		-0.0917015172		
$-K_4 \sin\beta_1 \sin\beta_2$	=	0.0367797603	=	D

$$E = -K_5 \sin\beta_1 \sin\beta_2$$

K_5		0.7994593686		
$\sin\beta_1 \sin\beta_2$		-0.0917015172		
$-K_5 \sin\beta_1 \sin\beta_2$	=	0.0733116370	=	E

$$F = CK_6$$

C		-0.0083372662		
K_6		3.9865206649		
CK_6	=	-0.0332366839	=	F

$$G = \theta^2/\sin\theta$$

θ		101°38'17".336		
θ (radians)		1.773920344		
θ^2		3.146793387		
$\theta^2/\sin\theta$	=	3.212846105	=	G

[illegible]

TABLE 23

COMPUTATION

PARAMETRIC LATITUDE & APPROXIMATE SPHERICAL DISTANCE

Parametric Latitude (β)

$$\tan \beta = (1-f) \tan \phi$$

f		0.0033670034	
$(1-f)$		0.9966329966	
$\tan \phi$	0.21558157		-0.48686087
$\tan \beta$	0.2148557061		-0.4852216078
β	$\beta_1 = 12^\circ\ 07'33''.521$		$\beta_2 = -25^\circ\ 53'01''.189$

APPROXIMATE SPHERICAL DISTANCE

$$\cos \theta = \sin \beta_1 \sin \beta_2 + \cos \beta_1 \cos \beta_2 \cos L$$

$L = \Delta \lambda$		$97^\circ\ 11'09''.145$	
$\sin L$		0.99214556	
$\cos L$		-0.12508862	
$\sin \beta$	0.2100618662		-0.4365452849
$\cos \beta$	0.9776880956		0.8996822851
$\sin \beta_1 \sin \beta_2$		-0.0917015172	
$\cos \beta_1 \cos \beta_2$		0.8796086600	
$\cos \beta_1 \cos \beta_2 \cos L$		-0.1100290334	
$\cos \theta$		-0.2017305506	

$$\sin \frac{\theta}{2} = \left[\cos \beta_1 \cos \beta_2 \sin^2 \frac{L}{2} + \sin^2 \frac{\beta_2 - \beta_1}{2} \right]^{\frac{1}{2}}$$

$L/2$	$48^\circ 35'34''.573$
$\sin \frac{L}{2}$	0.75002937
$\sin^2 L/2$	0.5625440559
$\beta_2 - \beta_1$	$-38^\circ 00'34''.710$
$(\beta_2 - \beta_1)/2$	$-19^\circ 00'17''.355$
$\sin (\beta_2 - \beta_1)/2$	-0.32564771
$\sin^2 (\beta_2 - \beta_1)/2$	0.1060464310
$\cos \beta_1 \cos \beta_2 \sin^2 L/2$	0.4948186232
$\left[\cos \beta_1 \cos \beta_2 \sin^2 \frac{L}{2} + \sin^2 \frac{\beta_2 - \beta_1}{2} \right]$	0.6008650542
$\sin \theta/2$	0.7751548582
$\theta/2$	$50^\circ 49'08''.668$
$\theta =$	$101^\circ 38'17''.336$
$\sin \theta$	0.9794410574
$\cos \theta$	-0.2017305506

[illegible]

TABLE 24

COMPUTATION OF $(\lambda - L)$, α_{12} and α_{21}.

$$(\lambda - L) = \left[\frac{(K_1 + B)\,\theta + (C + D)\sin\theta + (F + E)G}{K_2} \right] A\rho''$$

K_1	237.2388918
K_2	70519.51145
$(K_1 + B)$	238.0328052666
$(K_1 + B)\theta$	422.25123586
$(C + D)$	0.0284424941
$(C + D)\sin\theta$	0.0278477465
$(F + E)$	0.0400749531
$(F + E)\ G$	0.1287546570

$$(\lambda - L) = \left[\frac{422.4078483}{70519.51145} \right] (.8910182190)\ (\rho'') = 0^\circ\ 18'20''.866$$

$$\lambda = L + (\lambda - L) = 97^\circ\ 11'09''.145 + 0^\circ 18'20''.866 = 97^\circ\ 29'30''.011$$

$$\cot\alpha_{12} = \left[\tan\beta_2 \cos\beta_1 - \cos\lambda \sin\beta_1\right] / \sin\lambda$$

$$\cot\alpha_{12} = \frac{\left[(-0.4852216078)(.9776880956) - (-.1303820428)(0.2100618662)\right]}{0.9914638331}$$

$$\cot\alpha_{12} = -0.4470070945/0.9914638331 = -0.4508556737$$

$$\alpha_{12} = 114^\circ\ 16'\ 06''.607$$

$$\cot\alpha_{21} = \left[\sin\beta_2 \cos\lambda - \cos\beta_2 \tan\beta_1\right] / \sin\lambda$$

$$\cot\alpha_{21} = \frac{\left[(-.4365452849)(-0.1303820428) - (.8996822851)(.2148557061)\right]}{0.9914638331}$$

$$\cot\alpha_{21} = -0.1363842066/0.9914638331 = -0.1375584282$$

$$\alpha_{21} = 277^\circ\ 49'\ 56''.504$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

TABLE 25

COMPUTATION OF TRUE SPHERICAL DISTANCE, θ_o.

(USE λ VICE L)

COMPUTATION OF GEODESIC

$$\begin{aligned}
\cos\theta_o &= \sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos\lambda \\
&= (-.0917015172) + (.8796086600)(-.1303820428) \\
&= (-.0917015172) - (.114685174 0) \\
&= -.2063866912 \\
\theta_o &= 101^\circ\ 54'\ 38 \\
\theta_o &= 1.778677025 \text{ radians}
\end{aligned}$$

$$\cos\beta_o = \left[\cos\beta_1 \ \cos\beta_2 \sin\lambda\right] / \sin\theta_o$$

$\cos\beta_1 \cos\beta_2$	0.8796086600
$\sin\lambda$	0.9914638331
$\sin\theta_o$	0.9784705080
$\cos\beta_o$	0.8912891769 = maximum latitude of geodesic

$$A_o = 1 + (e'^2/4)\sin^2\beta_o - \left(\frac{3e'^4}{64}\right)\sin^4\beta_o$$

$e'^2/4$	0.0016920426
$\sin^2\beta_o$	0.2056036030
$3e'^4/64$	0.0000021473
$\sin^4\beta_o$	0.0422728416
A_o	1.0003477993

$$B_o = (e'^2/4)\sin^2\beta_o - (e'^4/16)\sin^4\beta_o$$

$e'^2/4$	0.0016920426
$\sin^2\beta_o$	0.2056036030
$e'^4/16$	0.0000028630
$\sin^4\beta_o$	0.0422728416
B_o	0.0003477691

$$C_o = \frac{e'^4}{128}\sin^4\beta$$

$e'^4/128$	0.0000003579
$\sin^4\beta_o$	0.0422728416
C_o	0.0000000151

$$\cos 2\sigma = \frac{2\sin\beta_1 \sin\beta_2}{\sin^2\beta_o} - \cos\theta_o$$

$2\sin\beta_1$	0.4201237324
$\sin\beta_2$	-0.4365452849
$\sin^2\beta_o$	0.2056036030
$\cos\theta_o$	0.2063866912
$\cos 2\sigma$	-0.6856357822
2σ	133°17'08".261
4σ	266°34'16".522
$\cos 4\sigma$	-0.0598071587

[illegible]

TABLE 25 (Continued)

$$s = b\,(A_o\,\theta_o + B_o \sin\theta_o \cos 2\sigma - C_o \sin 2\theta_o \cos 4\sigma)$$

A_o	1.0003477993	
θ_o	1.778677025	
$A_o\,\theta_o$	1.7792956471	
B_o	0.0003477691	
$\sin\theta_o$	0.9784705080	
$\cos 2\sigma$	-0.6856357822	
$B_o \sin\theta_o \cos 2\sigma$	-0.0002333094	
C_o	0.0000000151	
$\sin 2\theta_o$	-0.4038865814	
$\cos 4\sigma$	-0.0598071587	
$C_o \sin 2\theta_o \cos 4\sigma$	0.0000000004	
b	6,356,911.9462	
s	11,309,342.6228	meters
	6,102.4421	nautical miles
	7,027.5723	statute miles

7. CONCLUSION

Upon reviewing the results of the determination of the orbit of Explorer I, it can be stated that, in general, reliable orbit parameters cannot be reduced from only one set of three observations of the satellite.

Based on the results of the calculations of i and Ω, the position of the orbital plane in space can be accurately determined from a few observations. To determine the size and shape of the elliptical orbit and to accurately orient the orbit in its plane requires a series of precisely observed and reduced coordinates. As Moulton states [4, p. 44]: the difficulties arise in finding a, e, and w. As mentioned previously, these three parameters determine the size, shape, and orientation of the orbit in its plane. To determine these three parameters satisfactorily, the whole time interval of the three observations should be divided into two nearly equal parts by the second observation. If this is so then σ_1 and σ_3 will be nearly equal. With the time intervals equal or nearly so, the ratio of the area of the sector to the area of the triangles (see Figure 5) will be more reliable and the series which is used to determine the auxiliary parameter p will converge more rapidly. Additional terms can be used in the series to determine p.

In an analysis of the solution for u_1, u_2, and u_3 (derived from p), upon which σ_1, σ_2, and σ_3 are based, the reader should note the value of each u determined by its sine and cosine:

	u_1	u_2	u_3
sine value	248° 13'02".933	253° 13'50".317	255° 34'25".033
cosine value	248° 13'02".756	253° 13'50".305	255° 46'17".814
difference	0".177	0".012	11'52".781

There is a close agreement between the value of the angle determined by the sine and that determined by the cosine for u_1 and u_2 but there is an 11'52".781 disagreement between the value for u_3 derived from its sine and cosine. The value used for u_3 in the ensuing calculations was an average of these two which is 255° 40'21".424. This could be the source of the error existing between the computed a, e, and w and those calculated from the 82 observations. I believe that this can be attributed to the uneven time interval of the observations:

T_1	$22^h29^m10^s.140$)	
T_2	22 31 04. 420)	$1^m54^s.280$
T_3	22 32 03 .100)	$0^m58^s.680$

and the unrealistic observed declinations:

Obs. #1	Obs. #2	Obs. #3
47° 22' S	51° 06' S	49° 25' S

It seems logical to assume that the declination should either increase or decrease progressively during the short intervals of the observations, particularly during the same transit of the satellite. There is no such progression in this case.

The assumption that the orbit is correct is another source of error. The accuracy of the orbital elements predicted for the time of the observations from the unknown station depend on the accuracy of the initial values of the parameters determined from the 82 observations. The standard error of one observation of ± 5'.43 undoubtedly lessens the reliability of these values, especially the mean anomaly, M, which is probably the crux of the entire problem.

The assumption that the observed coordinates of the satellite and the computed orbital elements are referred to the same mean coordinate system also influences the result. As stated in [9, p.2] on the

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

orbital elements:

> the reference plane is defined as the true equator of the date. The origin of right ascension is a line shifted from the mean equinox of the date by an amount equal to the precession in right ascension between 1950.0 and the date.

Given below is a formula with which values can be obtained to correct the right ascension given in the orbital elements, in a right ascension referring to the mean equinox of the date:

$$\Omega^{o}(T) = \Omega^{o}(DOI) + 3^{o}.508 \times 10^{-5} \text{ (Smithsonian Day - 33286)}$$

where DOI indicates the values determined by the Differential Orbit Improvement Program. This information was not available.

It is apparent that the methods and procedures demonstrated herein to effect the tie between the two points on the surface of the earth can, with refinements, be employed in practical applications. The success of this application depends substantially on the characteristics of the satellite itself. In fact, the only satellite that would be practical for precise results would be a geodetic satellite — one that would have the following characteristics:

a. Have a nearly circular orbit or one that has a very small ellipticity - this would permit reliable predictions to be made in its position and eventually an ephemeris could be made available;

b. Be spherical in shape - so that its motion in space would not affect its presentation surface;

c. Have a perigee of approximately 1,000 miles to minimize the effect of atmospheric drag;

d. Have a large inclination angle so that it would be visible from most latitudes;

e. Have a flashing beacon that would be visible at its long range.

REFERENCES

[1] Beckner, L.V. and Odishaw, H.: Science in Space, McGraw-Hill Book Co., Inc., New York, New York, 1961.

[2] Brouwer, D. and Clemence, G.M.: Methods of Celestial Mechanics, Academic Press, Inc., New York, New York, 1961.

[3] Heiskanen, W.A. and Vening Meinesz, F.A.: The Earth and Its Gravity Field, McGraw-Hill Book Co., Inc., New York, New York, 1958.

[4] Moulton, F.R.: An Introduction to Celestial Mechanics, The Macmillan Co., New York, New York, 1960.

[5] Mueller, I.I.: Unpublished lecture notes, Physical Geodesy I, Geodetic Science 751, The Ohio State University, 1962.

[6] Rainsford, H.F.: Long Geodesics On the Ellipsoid, Bulletin Geodesique No. 37, 1955.

[7] Smithsonian Institution Astrophysical Observatory: Catalogue of Satellite Observations, Research in Space Science, Special Report No. 66. Cambridge, Massachusetts, 17 July 1961.

[8] Smithsonian Institution Astrophysical Observatory: List of Coordinates of Stations Engaged in the Observation of Artificial Earth Satellites, Research in Space Science, Special Report No. 69. Cambridge, Massachusetts, 17 July 1961.

[9] Smithsonian Institution Astrophysical Observatory: Satellite Orbital Data, Research in Space Science, Special Report No. 78. Cambridge, Massachusetts, 25 October 1961.

[10] Sterne, T.F.: Introduction to Celestial Mechanics, Interscience Publishers, Inc., New York, New York, 1960.

[11] Thomas, P.D.: Use of Near - Earth Satellite Orbits for Geodetic Information, U.S. Dept. of Commerce Technical Bulletin No. 11, 1960.

[12] U.S. Air Force: Determination of Ground Positions from Observations of Artificial Earth Satellites, Aeronautical Chart and Information Center Technical Report No. 86, St. Louis, Missouri. 1959

[13] U.S. Air Force: Geodetic Distance and Azimuth Computations for Lines over 500 Miles, Aeronautical Chart and Information Center Technical Report No. 80. St. Louis, Missouri. 1957.